Final Exam: MATH 410
Tuesday, 15 December 2015
Professor David Levermore

1. [10] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Give negations of each of the following assertions.

   (a) For every $\epsilon > 0$ there exists an $n_\epsilon \in \mathbb{N}$ such that
   $$m, n > n_\epsilon \implies |x_m - x_n| < \epsilon\,.$$

   (b) $\lim_{n\to\infty} x_n = -\infty\,.$

2. [20] State whether each of the following statements is true or false. Give a proof when true and a counterexample when false.

   (a) If the interval (a, b) is bounded, $f : (a, b) \to \mathbb{R}$ is differentiable, and $f' : (a, b) \to \mathbb{R}$ is bounded over (a, b) then the function f is bounded over (a, b).

   (b) If $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions such that each $f_n : [a, b] \to \mathbb{R}$ is Riemann integrable over $[a, b]$, and $f_n \to f$ pointwise over $[a, b]$ where $f : [a, b] \to \mathbb{R}$ is Riemann integrable over $[a, b]$, then
   $$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f\,.$$

3. [25] Consider a function g defined by
   $$g(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sin(kx)\,,$$
   for every $x \in \mathbb{R}$ for which the above series converges.

   (a) Show that g is defined for every $x \in \mathbb{R}$.

   (b) Show that g is continuously differentiable over $\mathbb{R}$ and that
   $$g'(x) = \sum_{k=1}^{\infty} \frac{k}{2^k} \cos(kx)\,.$$

4. [25] For every $n \in \mathbb{Z}_+$ define $f_n(x) = ne^{-nx}$ for every $x \in [0, 1]$.

   (a) Prove for every $\delta > 0$ that
   $$\lim_{n\to\infty} f_n = 0 \qquad \text{uniformly over } [\delta, 1]\,.$$

   (b) Show that
   $$\lim_{n\to\infty} \int_0^1 f_n = 1\,.$$

   (c) Let $g : [0, 1] \to \mathbb{R}$ be continuous. Show that
   $$\lim_{n\to\infty} \int_0^1 f_n g = g(0)\,.$$

More problems are on the back.

5. [20] Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be Riemann integrable over $[a, b]$. Prove that $f + g$ is Riemann integrable over $[a, b]$.

6. [20] For every $n \in \mathbb{Z}_+$ define $h_n(x) = nxe^{-nx}$ for every $x \in [0, 1]$.
   (a) Prove that $h_n \to 0$ pointwise over $[0, 1]$.
   (b) Prove that this limit is not uniform over $[0, 1]$.

7. [20] Determine all $a \in \mathbb{R}$ for which the following formal infinite series converge. Give your reasoning.
   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{a^n}{4^n \log(n)}$
   (b) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{k^2 + 1}\right)^a$

8. [20] Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous over (a, b). Let $\{x_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence contained in (a, b). Show that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence.

9. [20] Let $\alpha \in (0, 1]$ and $K \in \mathbb{R}_+$ such that the function $f : [a, b] \to \mathbb{R}$ satisfy the Hölder bound
$$|f(x) - f(y)| < K\,|x - y|^\alpha \qquad \text{for every } x, y \in [a, b]\,.$$
   (a) Show that f is uniformly continuous over $[a, b]$.
   (b) Show that for every partition P of $[a, b]$ one has
$$0 \le U(f, P) - L(f, P) < |P|^\alpha K\,(b - a)\,.$$

10. [20] Show that
$$\log(1 + x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\, x^{2k} \quad \text{for every } x \in [-1, 1]\,.$$
Hint: Consider $\log(1 + y)$ first.