Third In-Class Exam Solutions
Math 246, Professor David Levermore
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(1) [10] The vertical displacement of an unforced mass on a spring is given by

$$h(t) = -5e^{-3t}\cos(4t) - 12e^{-3t}\sin(4t)\,.$$

(a) [2] Is this system undamped, under damped, critically damped, or over damped? (Give your reasoning!)
(b) [5] Express $h(t)$ in the amplitude-phase form $h(t) = Ae^{-3t}\cos(4t-\delta)$ with $A > 0$ and $0 \le \delta < 2\pi$. Label the amplitude and phase. (The phase may be expressed in terms of an inverse trig function.)
(c) [3] Give the natural frequency and natural period of this spring-mass system.

Solution (a). The system is *under damped* because the vertical displacement $h(t)$ arises from a characteristic polynomial with the conjugate pair of roots $-3 \pm i4$.

Alternative Solution (a). The system is *under damped* because the displacement $h(t)$ is a decaying oscillation, which is evident from the decaying exponential e^{-3t} multiplying the oscillatory trigonometric functions $\cos(4t)$ and $\sin(4t)$.

Remark. Both the e^{-3t} and the $\cos(4t)$ and $\sin(4t)$ must play a role in your reasoning for full credit!

Solution (b). By comparing

$$Ae^{-3t}\cos(4t-\delta) = Ae^{-3t}\cos(\delta)\cos(4t) + Ae^{-3t}\sin(\delta)\sin(4t)\,,$$

with $h(t) = -5e^{-3t}\cos(4t) - 12e^{-3t}\sin(4t)$, we see that

$$A\cos(\delta) = -5\,, \qquad A\sin(\delta) = -12\,.$$

This shows that (A,δ) are the polar coordinates of the point in the plane whose Cartesian coordinates are $(-5,-12)$. Clearly A is given by

$$A = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25+144} = \sqrt{169} = 13\,.$$

Because $(-5,-12)$ lies in the *third quadrant*, the phase δ must satisfy $\pi < \delta < \frac{3}{2}\pi$. We can express δ several ways. A picture shows that if we use π as a reference then

$$\cos(\delta-\pi) = \tfrac{5}{13}\,, \qquad \sin(\delta-\pi) = \tfrac{12}{13}\,, \qquad \tan(\delta-\pi) = \tfrac{12}{5}\,,$$

whereby we can express the phase by any one of the formulas

$$\delta = \pi + \cos^{-1}\!\left(\tfrac{5}{13}\right), \qquad \delta = \pi + \sin^{-1}\!\left(\tfrac{12}{13}\right), \qquad \delta = \pi + \tan^{-1}\!\left(\tfrac{12}{5}\right).$$

The same picture shows that if we use $\frac{3}{2}\pi$ as a reference then

$$\cos\!\left(\tfrac{3}{2}\pi - \delta\right) = \tfrac{12}{13}\,, \qquad \sin\!\left(\tfrac{3}{2}\pi - \delta\right) = \tfrac{5}{13}\,, \qquad \tan\!\left(\tfrac{3}{2}\pi - \delta\right) = \tfrac{5}{12}\,,$$

whereby we can express the phase by any one of the formulas

$$\delta = \tfrac{3}{2}\pi - \cos^{-1}\!\left(\tfrac{12}{13}\right), \qquad \delta = \tfrac{3}{2}\pi - \sin^{-1}\!\left(\tfrac{5}{13}\right), \qquad \delta = \tfrac{3}{2}\pi - \tan^{-1}\!\left(\tfrac{5}{12}\right).$$

Only one expression for δ is required.

Remark. It is incorrect to give the phase by one of the formulas

$$\delta = \cos^{-1}\!\left(-\tfrac{5}{13}\right), \qquad \delta = \sin^{-1}\!\left(-\tfrac{12}{13}\right), \qquad \delta = \tan^{-1}\!\left(\tfrac{12}{5}\right),$$

because, by our conventions for the range of the inverse trigonometric functions, $\cos^{-1}\big(-\frac{5}{13}\big)$ lies in $(\frac{\pi}{2},\pi)$, $\sin^{-1}\big(-\frac{12}{13}\big)$ lies in $(-\frac{\pi}{2},0)$, and $\tan^{-1}\big(\frac{12}{5}\big)$ lies in $(0,\frac{\pi}{2})$.

Solution (c). Because the underlying characteristic polynomial has the conjugate pair of roots $-3 \pm i4$, it must be

$$p(z) = (z+3)^2 + 4^2 = z^2 + 6z + 9 + 16 = z^2 + 6z + 25\,.$$

Therefore the vertical displacement $h(t)$ satisfies the differential equation

$$\ddot{h} + 6\dot{h} + 25h = 0\,.$$

We can read off that the natural frequency is $\omega_o = \sqrt{25} = 5$ radians per sec, whereby the natural period T_o is given by

$$T_o = \frac{2\pi}{\omega_o} = \frac{2\pi}{5} \quad \text{sec}\,.$$

(2) [6] When a 10 gram mass is hung vertically from a spring, at rest it stretches the spring 5.0 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) The medium imparts a damping force of 160 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 2 cm/sec. At $t = 0$ the mass is displaced 3 cm below its rest position and is released with a upward velocity of 2 cm/sec. Assume that the spring force is proportional to displacement, that the damping is proportional to veloicity, and that there are no other forces. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

Solution. Let $h(t)$ be the displacement (in centimeters) of the mass from its rest position at time t (in seconds), with upward displacements being positive. The governing initial-value problem then has the form

$$m\ddot{h} + \gamma\dot{h} + kh = 0\,, \qquad h(0) = -3\,, \quad \dot{h}(0) = 2\,,$$

where m is the mass, γ is the damping coefficient, and k is the spring constant. We are given that $m = 10$ grams. We obtain k by balancing the force applied by the spring when it is stretched 5.0 cm with the weight of the mass ($mg = 10 \cdot 980$ dynes). This gives $k\,5.0 = 10 \cdot 980$, or

$$k = \frac{10 \cdot 980}{5.0} = 2 \cdot 980 \text{ dynes/cm}\,.$$

We obtain γ by balancing the damping force when the speed of the mass is 2 cm/sec with 160 dynes. This gives $\gamma\,2 = 160$, or

$$\gamma = \frac{160}{2} \quad \text{dynes sec/cm}\,.$$

Therefore the governing initial-value problem is

$$10\ddot{h} + \tfrac{160}{2}\dot{h} + 2 \cdot 980h = 0\,, \qquad h(0) = -3\,, \quad \dot{h}(0) = 2\,.$$

Remark. Had we chosen the convention of downward displacements being positive then the governing initial-value problem is

$$10\ddot{h} + \tfrac{160}{2}\dot{h} + 2 \cdot 980h = 0\,, \qquad h(0) = 3\,, \quad \dot{h}(0) = -2\,.$$

(3) [6] Recast the ordinary differential equation $v'''' = \sin(v)v''' + v^3 v'' + t^2\cos(v')$ as a first-order system of ordinary differential equations.

Solution. Because the equation is fourth order, the first-order system must have dimension at least four. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \sin(x_1)x_4 + x_1^3 x_3 + t^2\cos(x_2) \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} v \\ v' \\ v'' \\ v''' \end{pmatrix}.$$

(4) [12] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^2 \\ -1 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$.

(a) [2] Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [4] Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) [2] Give a general solution to the system that you found in part (b).

(d) [4] Find the natural fundamental matrix associated with the initial time 0 for the system that you found in part (b).

Solution (a). The Wronskian is

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^2 & e^{-t} \\ -1 & e^{-t} \end{pmatrix} = t^2 \cdot e^{-t} - (-1)\cdot e^{-t} = (t^2+1)e^{-t}.$$

Solution (b). Let $\mathbf{\Psi}(t) = \begin{pmatrix} t^2 & e^{-t} \\ -1 & e^{-t} \end{pmatrix}$. Because $\mathbf{\Psi}'(t) = \mathbf{A}(t)\mathbf{\Psi}(t)$, we have

$$\begin{aligned} \mathbf{A}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 2t & -e^{-t} \\ 0 & -e^{-t} \end{pmatrix}\begin{pmatrix} t^2 & e^{-t} \\ -1 & e^{-t} \end{pmatrix}^{-1} \\ &= \frac{1}{(1+t^2)e^{-t}}\begin{pmatrix} 2t & -e^{-t} \\ 0 & -e^{-t} \end{pmatrix}\begin{pmatrix} e^{-t} & -e^{-t} \\ 1 & t^2 \end{pmatrix} \\ &= \frac{1}{(1+t^2)e^{-t}}\begin{pmatrix} 2te^{-t} - e^{-t} & -2te^{-t} - t^2 e^{-t} \\ -e^{-t} & -t^2 e^{-t} \end{pmatrix} \\ &= \frac{1}{1+t^2}\begin{pmatrix} 2t-1 & -t^2-2t \\ -1 & -t^2 \end{pmatrix}. \end{aligned}$$

Solution (c). A general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{pmatrix} t^2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}.$$

Solution (d). By using the fundamental matrix $\mathbf{\Psi}(t)$ from part (b) we find that the natural fundamental matrix associated with the initial time 0 is

$$\begin{aligned} \mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} &= \begin{pmatrix} t^2 & e^{-t} \\ -1 & e^{-t} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} t^2 & e^{-t} \\ -1 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} t^2 + e^{-t} & -t^2 \\ e^{-t} - 1 & 1 \end{pmatrix}. \end{aligned}$$

(5) [8] Find a general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & -3 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 2z - 15 = (z-3)(z+5).$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 3 and -5. These can be expressed as -1 ± 4. Then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-t}\left[\cosh(4t)\mathbf{I} + \frac{\sinh(4t)}{4}\big(\mathbf{A} - (-1)\mathbf{I}\big)\right] \\ &= e^{-t}\left[\cosh(4t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(4t)}{4}\begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix}\right] \\ &= e^{-t}\begin{pmatrix} \cosh(4t) + \frac{1}{2}\sinh(4t) & \sinh(4t) \\ \frac{3}{4}\sinh(4t) & \cosh(4t) - \frac{1}{2}\sinh(4t) \end{pmatrix}. \end{aligned}$$

(Check that $\mathbf{A} - (-1)\mathbf{I}$ has trace zero!) Therefore a general solution of the system is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} = c_1\begin{pmatrix} \cosh(4t) + \frac{1}{2}\sinh(4t) \\ \frac{3}{4}\sinh(4t) \end{pmatrix} + c_2\begin{pmatrix} \sinh(4t) \\ \cosh(4t) - \frac{1}{2}\sinh(4t) \end{pmatrix}.$$

(6) [8] Find a general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 4 & 4 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 4z + 4 = (z-2)^2.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is only 2. Then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{2t}\left[\mathbf{I} + t\,(\mathbf{A} - 2\mathbf{I})\right] \\ &= e^{2t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\right] = e^{2t}\begin{pmatrix} 1-2t & -t \\ 4t & 1+2t \end{pmatrix}, \end{aligned}$$

(Check that $\mathbf{A} - 2\mathbf{I}$ has trace zero!) Therefore a general solution of the system is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} = c_1 e^{2t}\begin{pmatrix} 1-2t \\ 4t \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} -t \\ 1+2t \end{pmatrix}.$$

(7) [10] Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 5 = (z-1)^2 + 2^2\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 + i2$ and $1 - i2$. Then

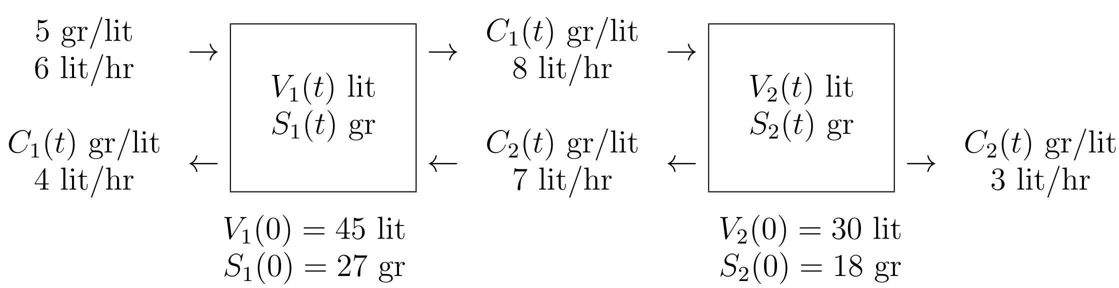
$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}\,(\mathbf{A} - 1\mathbf{I}) \right] \\ &= e^t \left[\cos(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2} \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix}\,. \end{aligned}$$

(Check that $\mathbf{A} - \mathbf{I}$ has trace zero!) Therefore a general solution of the initial-value problem is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^I = e^t \begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = e^t \begin{pmatrix} 3\cos(2t) + \frac{3}{2}\sin(2t) \\ -6\sin(2t) + 3\cos(2t) \end{pmatrix}\,.$$

(8) [6] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 45 liters and the second contains 30 liters. Brine with a salt concentration of 5 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 8 liters per hour, from the second into the first at 7 liters per hour, from the first into a drain at 4 liter per hour, and from the second into a drain at 3 liters per hour. At $t = 0$ there are 27 grams of salt in the first tank and 18 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t hours. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t hours. Because mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the concentrations of the brine that flows out of these tanks. We have the following picture.



We are asked to write down an initial-value problem that governs $S_1(t)$ and $S_2(t)$.

The rates work out so there will be $V_1(t) = 45 + t$ liters of brine in the first tank and $V_2(t) = 30 - 2t$ liters in the second. Then $S_1(t)$ and $S_2(t)$ are governed by the initial-value problem

$$\begin{aligned}
\frac{\mathrm{d}S_1}{\mathrm{d}t} &= 5\cdot 6 + \frac{S_2}{30-2t}\,7 - \frac{S_1}{45+t}\,8 - \frac{S_1}{45+t}\,4\,, && S_1(0) = 27\,,\\
\frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{45+t}\,8 - \frac{S_2}{30-2t}\,7 - \frac{S_2}{30-2t}\,3\,, && S_2(0) = 18\,.
\end{aligned}$$

You could leave the answer in the above form. However, it can be simplified to

$$\begin{aligned}
\frac{\mathrm{d}S_1}{\mathrm{d}t} &= 30 + \frac{7}{30-2t}\,S_2 - \frac{12}{45+t}\,S_1\,, && S_1(0) = 27\,,\\
\frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{8}{45+t}\,S_1 - \frac{5}{15-t}\,S_2\,, && S_2(0) = 18\,.
\end{aligned}$$

Notice that the interval of definition for this initial-value problem is $(-45, 15)$.

(9) [12] Consider the following MATLAB commands.

```
>> syms t s Y; f = ['t^3 + heaviside(t - 2)*(8 - t^3)'];
>> diffeqn = sym('D(D(y))(t) - 6*D(y)(t) + 13*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s)', 'y(0)', 'D(y)(0)'}, {Y, -2, 5});
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) [4] Give the initial-value problem for $y(t)$ that is being solved.
(b) [8] Find the Laplace transform $Y(s)$ of the solution $y(t)$.

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution (a). The initial-value problem for $y(t)$ that is being solved is

$$y'' - 6y' + 13y = f(t)\,, \qquad y(0) = -2\,, \quad y'(0) = 5\,,$$

where the forcing $f(t)$ can be expressed either as

$$f(t) = \begin{cases} t^3 & \text{for } 0 \le t < 2\,,\\ 8 & \text{for } 2 \le t\,, \end{cases}$$

or in terms of the unit step function as $f(t) = t^3 + u(t-2)(8-t^3)$.

Solution (b). The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) - 6\mathcal{L}[y'](s) + 13\mathcal{L}[y](s) = \mathcal{L}[f](s)\,.$$

Because

$$\begin{aligned}
\mathcal{L}[y](s) &= Y(s)\,,\\
\mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) + 2\,,\\
\mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + 2s - 5\,,
\end{aligned}$$

the Laplace transform of the initial-value problem becomes

$$\big(s^2Y(s) + 2s - 5\big) - 6\big(sY(s) + 2\big) + 13Y(s) = \mathcal{L}[f](s)\,.$$

This simplifies to

$$(s^2 - 6s + 13)Y(s) + 2s - 17 = \mathcal{L}[f](s)\,,$$

whereby

$$Y(s) = \frac{1}{s^2 - 6s + 13}\big(-2s + 17 + \mathcal{L}[f](s)\big)\,.$$

To compute $\mathcal{L}[f](s)$, we write $f(t)$ as

$$f(t) = t^3 + u(t-2)(8 - t^3) = t^3 + u(t-2)j(t-2)\,,$$

where by setting $j(t-2) = 8 - t^3$ we see that

$$j(t) = 8 - (t+2)^3 = 8 - \big(t^3 + 6t^2 + 12t + 8\big) = -t^3 - 6t^2 - 12t\,.$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 3$, with $a = 0$ and $n = 2$, and with $a = 0$ and $n = 1$ shows that

$$\mathcal{L}[t^3](s) = \frac{6}{s^4}\,, \qquad \mathcal{L}[t^2](s) = \frac{2}{s^3}\,, \qquad \mathcal{L}[t](s) = \frac{1}{s^2}\,,$$

whereby item 6 with $c = 2$ and $j(t) = -t^3 - 6t^2 - 12t$ shows that

$$\begin{aligned}\mathcal{L}\big[u(t-2)j(t-2)\big](s) = e^{-2s}\mathcal{L}[j](s) &= -e^{-2s}\mathcal{L}\big[t^3 + 6t^2 + 12t\big](s)\\ &= -e^{-2s}\left(\frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2}\right)\,.\end{aligned}$$

Therefore

$$\mathcal{L}[f](s) = \mathcal{L}\big[t^3 + u(t-2)j(t-2)\big](s) = \frac{6}{s^4} - e^{-2s}\left(\frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2}\right)\,.$$

Upon placing this result into the expression for $Y(s)$ found earlier, we obtain

$$Y(s) = \frac{1}{s^2 - 6s + 13}\left(-2s + 17 + \frac{6}{s^4} - e^{-2s}\left(\frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2}\right)\right)\,.$$

(10) [6] Compute the Green function $g(t)$ for the differential operator $(\mathrm{D} + 4)^3$ where $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$.

Solution. The operator $(\mathrm{D} + 4)^3$ has characteristic polynomial $p(s) = (s + 4)^3$. Therefore its Green function $g(t)$ is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{(s+4)^3}\right](t)\,.$$

Referring to the table on the last page, item 1 with $a = -4$ and $n = 2$ gives

$$g(t) = \tfrac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+4)^3}\right](t) = \tfrac{1}{2}t^2 e^{-4t}\,.$$

(11) [8] Compute the Laplace transform of $f(t) = u(t-4)\, e^{-2t}$ from its definition. (Here u is the unit step function.)

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} u(t-4)\, e^{-2t}\, \mathrm{d}t = \lim_{T\to\infty} \int_4^T e^{-(s+2)t}\, \mathrm{d}t\,.$$

When $s \le -2$ this limit diverges to $+\infty$ because in that case we have for every $T > 4$

$$\int_4^T e^{-(s+2)t}\, \mathrm{d}t \ge \int_4^T \mathrm{d}t = T - 4\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

When $s > -2$ we have for every $T > 4$

$$\int_4^T e^{-(s+2)t}\, \mathrm{d}t = -\frac{e^{-(s+2)t}}{s+2}\bigg|_4^T = -\frac{e^{-(s+2)T}}{s+2} + \frac{e^{-(s+2)4}}{s+2}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \left[-\frac{e^{-(s+2)T}}{s+2} + \frac{e^{-(s+2)4}}{s+2} \right] = \frac{e^{-(s+2)4}}{s+2} \qquad \text{for } s > -2\,.$$

(12) [8] Find the inverse Laplace transform $\mathcal{L}^{-1}[Y(s)](t)$ of the function

$$Y(s) = e^{-3s}\, \frac{3s+13}{s^2 - 3s - 4}\,.$$

You may refer to the table on the last page.

Solution. Referring to the table on the last page, item 6 with $c = 3$ implies that

$$\mathcal{L}^{-1}\big[e^{-3s}\, J(s)\big] = u(t-3) j(t-3)\,, \qquad \text{where} \qquad j(t) = \mathcal{L}^{-1}[J(s)](t)\,.$$

We apply this formula to

$$J(s) = \frac{3s+13}{s^2 - 3s - 4}\,.$$

Because the denominator factors as $(s-4)(s+1)$, we have the partial fraction identity

$$\frac{3s+13}{s^2-3s-4} = \frac{3s+13}{(s-4)(s+1)} = \frac{5}{s-4} + \frac{-2}{s+1}\,.$$

Referring to the table on the last page, item 1 with $a = 4$ and $n = 0$, and with $a = -1$ and $n = 0$ implies that

$$\mathcal{L}^{-1}\left[\frac{1}{s-4}\right](t) = e^{4t}\,, \qquad \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = e^{-t}\,.$$

These formulas also can be obtained from item 2 with $a = 4$ and $b = 0$, and with $a = -1$ and $b = 0$.

The above formulas and the linearity of the inverse Laplace transform yield

$$\begin{aligned} j(t) = \mathcal{L}^{-1}[J(s)](t) &= \mathcal{L}^{-1}\left[\frac{3s+13}{s^2-3s-4}\right](t) \\ &= \mathcal{L}^{-1}\left[\frac{5}{s-4} + \frac{-2}{s+1}\right](t) \\ &= 5\mathcal{L}^{-1}\left[\frac{1}{s-4}\right](t) - 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = 5e^{4t} - 2e^{-t}\,. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}^{-1}\big[Y(s)\big](t) = \mathcal{L}^{-1}[e^{-3s}J(s)](t) &= u(t-3)j(t-3) \\ &= u(t-3)\left(5e^{4(t-3)} - 2e^{-(t-3)}\right). \end{aligned}$$

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)$$

and u is the unit step function.