Solutions of Sample Problems for Third In-Class Exam
Math 246, Spring 2017, Professor David Levermore

(1) The vertical displacement of a mass on a spring is given by

$$h(t) = 4e^{-t}\cos(7t) - 3e^{-t}\sin(7t)\,,$$

where positive displacements are upward.

(a) Express $h(t)$ in the form $h(t) = Ae^{-t}\cos(\omega t - \delta)$ with $A > 0$ and $0 \leq \delta < 2\pi$, identifying the quasiperiod and phase of the oscillation. (The phase may be expressed in terms of an inverse trig function.)

(b) Sketch the solution over $0 \leq t \leq 2$.

Solution (a). By comparing

$$Ae^{-t}\cos(\omega t - \delta) = Ae^{-t}\cos(\delta)\cos(\omega t) + Ae^{-t}\sin(\delta)\sin(\omega t)\,,$$

with $h(t) = 4e^{-t}\cos(7t) - 3e^{-t}\sin(7t)$, we see that $\omega = 7$ and that

$$A\cos(\delta) = 4\,, \qquad A\sin(\delta) = -3\,.$$

This shows that (A, δ) are the polar coordinates of the point in the plane whose Cartesian coordinates are $(4, -3)$. Clearly A is given by

$$A = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5\,.$$

Because $(4, -3)$ lies in the fourth quadrant, the phase δ satisfies $\frac{3\pi}{2} < \delta < 2\pi$. There are several ways to express δ. A picture shows that if we use 2π as a reference then

$$\sin(2\pi - \delta) = \tfrac{3}{5}\,, \qquad \tan(2\pi - \delta) = \tfrac{3}{4}\,, \qquad \cos(2\pi - \delta) = \tfrac{4}{5}\,,$$

and we can express the phase by any one of the formulas

$$\delta = 2\pi - \sin^{-1}\!\left(\tfrac{3}{5}\right), \qquad \delta = 2\pi - \tan^{-1}\!\left(\tfrac{3}{4}\right), \qquad \delta = 2\pi - \cos^{-1}\!\left(\tfrac{4}{5}\right).$$

Finally, because the quasifrequency is $\omega = 7$, the quasiperiod T is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{7}\,.$$

Solution (b). This will be shown during the review session if someone asks for it.

(2) When a mass of 4 grams is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) At $t = 0$ the mass is displaced 3 cm above its equilibrium position and is released with no initial velocity. It moves in a medium that imparts a drag force of 2 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. There are no other forces. (Assume that the spring force is proportional to displacement and that the drag force is proportional to velocity.)

(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

(b) What is the natural frequency of the spring?

(c) Show that the system is under damped and find its quasifrequency.

Solution (a). Let $h(t)$ be the displacement of the mass from its equilibrium (rest) position at time t in centimeters, with upward displacements being positive. The governing initial-value problem then has the form

$$m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}h}{\mathrm{d}t} + kh = 0\,, \qquad h(0) = 3\,, \quad h'(0) = 0\,,$$

where m is the mass, γ is the drag coefficient, and k is the spring constant. The problem says that $m = 4$ grams. The spring constant is obtained by balancing the weight of the mass ($mg = 4 \cdot 980$ dynes) with the force applied by the spring when it is stetched 9.8 cm. This gives $k\,9.8 = 4 \cdot 980$, or

$$k = \frac{4 \cdot 980}{9.8} = 400 \quad \text{dynes/cm}\,.$$

The drag coefficient is obtained by balanceing the force of 2 dynes with the drag force imparted by the medium when the speed of the mass is 4 cm/sec. This gives $\gamma\,4 = 2$, or

$$\gamma = \frac{2}{4} = \frac{1}{2} \quad \text{dynes sec/cm}\,.$$

Therefore the governing initial-value problem is

$$4\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + \frac{1}{2}\frac{\mathrm{d}h}{\mathrm{d}t} + 400h = 0\,, \qquad h(0) = 3\,, \quad h'(0) = 0\,,$$

If we had chosen downward displacements to be positive then the governing initial-value problem would be the same except for the first initial condition, which would then be $h(0) = -3$.

Solution (b). The natural frequency of the spring is given by

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \cdot 980}{4 \cdot 9.8}} = \sqrt{100} = 10 \quad 1/\text{sec}\,.$$

Solution (c). The characteristic polynomial is

$$p(z) = z^2 + \frac{1}{8}z + 100 = \left(z + \frac{1}{16}\right)^2 + 100 - \frac{1}{16^2}\,,$$

which has a conjugate pair of roots. Therefore the system is under damped. The roots are $-\frac{1}{16} \pm i\nu$ where

$$\nu = \sqrt{100 - \frac{1}{16^2}} \quad 1/\text{sec}\,.$$

This is the quasifrequency.

(3) Compute the Laplace transform of $f(t) = t\,e^{3t}u(t-2)$ from its definition.

Solution. The definition of the Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\int_0^T e^{-st}t\,e^{3t}u(t-2)\,\mathrm{d}t = \lim_{T\to\infty}\int_2^T t\,e^{-(s-3)t}\,\mathrm{d}t\,.$$

This limit diverges to $+\infty$ for $s \leq 3$ because in that case for every $T > 2$ we have

$$\int_2^T t\, e^{-(s-3)t}\, \mathrm{d}t \geq \int_2^T t\, \mathrm{d}t = \frac{T^2}{2} - 2\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

For $s > 3$ an integration by parts shows that

$$\begin{aligned}
\int_2^T t\, e^{-(s-3)t}\, \mathrm{d}t &= -t\, \frac{e^{-(s-3)t}}{s-3}\bigg|_2^T + \int_2^T \frac{e^{-(s-3)t}}{s-3}\, \mathrm{d}t \\
&= \left(-\, t\, \frac{e^{-(s-3)t}}{s-3} - \frac{e^{-(s-3)t}}{(s-3)^2} \right)\bigg|_2^T \\
&= \left(-\, T\, \frac{e^{-(s-3)T}}{s-3} - \frac{e^{-(s-3)T}}{(s-3)^2} \right) + \left(2\, \frac{e^{-(s-3)2}}{s-3} + \frac{e^{-(s-3)2}}{(s-3)^2} \right).
\end{aligned}$$

Hence, for $s > 3$ we have that

$$\begin{aligned}
\mathcal{L}[f](s) &= \lim_{T\to\infty} \left[\left(-\, T\, \frac{e^{-(s-3)T}}{s-3} - \frac{e^{-(s-3)T}}{(s-3)^2} \right) + \left(2\, \frac{e^{-(s-3)2}}{s-3} + \frac{e^{-(s-3)2}}{(s-3)^2} \right) \right] \\
&= \frac{e^{-(s-3)2}}{(s-3)^2} + 2\, \frac{e^{-(s-3)2}}{s-3} - \lim_{T\to\infty} \left(T\, \frac{e^{-(s-3)T}}{s-3} + \frac{e^{-(s-3)T}}{(s-3)^2} \right) \\
&= \frac{e^{-(s-3)2}}{(s-3)^2} + 2\, \frac{e^{-(s-3)2}}{s-3}\,.
\end{aligned}$$

(4) Consider the following MATLAB commands.

```
>> syms t s Y; f = ['heaviside(t)*t^2 + heaviside(t − 3)*(3*t − t^2)'];
>> diffeqn = sym('D(D(y))(t) − 6*D(y)(t) + 10*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s)', 'y(0)', 'D(y)(0)'}, {Y, 2, 3});
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) Give the initial-value problem for $y(t)$ that is being solved.
(b) Find the Laplace transform $Y(s)$ of the solution $y(t)$.

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution (a). The initial-value problem for $y(t)$ that is being solved is

$$y'' - 6y' + 10y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 3\,,$$

where the forcing $f(t)$ can be expressed either as

$$f(t) = \begin{cases} t^2 & \text{for } 0 \leq t < 3\,, \\ 3t & \text{for } 3 \leq t\,, \end{cases}$$

or in terms of the unit step function as $f(t) = t^2 + u(t-3)(3t - t^2)$.

Solution (b). The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) - 6\mathcal{L}[y'](s) + 10\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned}\mathcal{L}[y](s) &= Y(s)\,,\\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 2\,,\\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 3\,.\end{aligned}$$

To compute $\mathcal{L}[f](s)$, we first write $f(t)$ as

$$f(t) = t^2 + u(t-3)(3t - t^2) = t^2 + u(t-3)j(t-3)\,,$$

where by setting $j(t-3) = 3t - t^2$ we see that

$$j(t) = 3(t+3) - (t+3)^2 = 3t + 9 - t^2 - 6t - 9 = -t^2 - 3t\,.$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 2$ and with $a = 0$ and $n = 1$ shows that

$$\mathcal{L}[t^2](s) = \frac{2}{s^3}\,, \qquad \mathcal{L}[t](s) = \frac{1}{s^2}\,,$$

whereby item 6 with $c = 3$ and $j(t) = -t^2 - 3t$ shows that

$$\mathcal{L}\big[u(t-3)j(t-3)\big](s) = e^{-3s}\mathcal{L}[j](s) = -e^{-3s}\mathcal{L}\big[t^2 + 3t\big](s) = -e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

Therefore

$$\mathcal{L}[f](s) = \mathcal{L}\big[t^2 + u(t-3)j(t-3)\big](s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 2s - 3\big) - 6\big(sY(s) - 2\big) + 10Y(s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right),$$

which becomes

$$(s^2 - 6s + 10)Y(s) - 2s + 9 = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

Therefore $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 - 6s + 10}\left(2s - 9 + \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right)\right).$$

(5) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$y'' + 4y' + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases}\cos(t) & \text{for } 0 \le t < 2\pi\,,\\ t - 2\pi & \text{for } t \ge 2\pi\,.\end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 13\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned}
\mathcal{L}[y](s) &= Y(s)\,,\\
\mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 4\,,\\
\mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 4s - 1\,.
\end{aligned}$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned}
f(t) &= \big(1 - u(t-2\pi)\big)\cos(t) + u(t-2\pi)(t-2\pi)\\
&= \cos(t) - u(t-2\pi)\cos(t) + u(t-2\pi)(t-2\pi)\\
&= \cos(t) - u(t-2\pi)\cos(t-2\pi) + u(t-2\pi)(t-2\pi)\,.
\end{aligned}$$

Referring to the table on the last page, item 6 with $c = 2\pi$ and $j(t) = \cos(t)$, item 6 with $c = 2\pi$ and $j(t) = t$, item 2 with $a = 0$ and $b = 1$, and item 1 with $n = 1$ and $a = 1$ then show that

$$\begin{aligned}
\mathcal{L}[f](s) &= \mathcal{L}[\cos(t)](s) - \mathcal{L}[u(t-2\pi)\cos(t-2\pi)](s) + \mathcal{L}[u(t-2\pi)(t-2\pi)](s)\\
&= \mathcal{L}[\cos(t)](s) - e^{-2\pi s}\mathcal{L}[\cos(t)](s) + e^{-2\pi s}\mathcal{L}[t](s)\\
&= \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,.
\end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 4s - 1\big) + 4\big(sY(s) - 4\big) + 13Y(s) = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,,$$

which becomes

$$(s^2 + 4s + 13)Y(s) - 4s - 1 - 16 = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2+4s+13}\left(4s + 17 + \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\right).$$

(6) Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

(a) $F(s) = \dfrac{2}{(s+5)^2}\,,$

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}\,,$

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}\,.$

Solution (a). Referring to the table on the last page, item 1 with $n = 1$ and $a = -5$ gives

$$\mathcal{L}[t\,e^{-5t}](s) = \frac{1}{(s+5)^2}\,.$$

Therefore we conclude that

$$\mathcal{L}^{-1}\left[\frac{2}{(s+5)^2}\right](t) = 2\mathcal{L}^{-1}\left[\frac{1}{(s+5)^2}\right](t) = 2t\,e^{-5t}\,.$$

Solution (b). Because the denominator factors as $(s-3)(s+2)$, we have the partial fraction identity

$$\frac{3s}{s^2-s-6} = \frac{3s}{(s-3)(s+2)} = \frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}\,.$$

Referring to the table on the last page, item 1 with $n = 0$ and $a = 3$, and with $n = 0$ and $a = -2$ gives

$$\mathcal{L}[e^{3t}](s) = \frac{1}{s-3}\,, \qquad \mathcal{L}[e^{-2t}](s) = \frac{1}{s+2}\,.$$

Therefore we conclude that

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{3s}{s^2-s-6}\right](t) &= \mathcal{L}^{-1}\left[\frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}\right](t)\\ &= \tfrac{9}{5}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right](t) + \tfrac{6}{5}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t)\\ &= \tfrac{9}{5}e^{3t} + \tfrac{6}{5}e^{-2t}\,.\end{aligned}$$

Solution (c). Complete the square in the denominator to get $(s-2)^2+1$. Referring to the table on the last page, item 2 with $a = 2$ and $b = 1$ gives

$$\mathcal{L}[e^{2t}\cos(t)](s) = \frac{s-2}{(s-2)^2+1}\,.$$

Item 6 with $c = 3$ and $j(t) = e^{2t}\cos(t)$ then gives

$$\mathcal{L}[u(t-3)e^{2(t-3)}\cos(t-3)](s) = e^{-3s}\frac{s-2}{(s-2)^2+1}\,.$$

Therefore we conclude that

$$\mathcal{L}^{-1}\left[e^{-3s}\frac{s-2}{s^2-4s+5}\right](t) = u(t-3)e^{2(t-3)}\cos(t-3)\,.$$

(7) Compute the Green function $g(t)$ for the following differential operators.

(a) $\mathrm{L} = (\mathrm{D}-2)^3\,,$

(b) $\mathrm{L} = \mathrm{D}^4 + 8\mathrm{D}^2 - 9\,.$

Solution (a). The characteristic polynomial of $\mathrm{L} = (\mathrm{D}-2)^3$ is $p(s) = (s-2)^3$. Therefore its Green function $g(t)$ is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{(s-2)^3}\right](t)\,.$$

Referring to the table on the last page, item 1 with $a = 2$ and $n = 2$ gives

$$g(t) = \tfrac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s-2)^3}\right] = \tfrac{1}{2}t^2 e^{2t}\,.$$

Solution (b). The characteristic polynomial of $\mathrm{L} = \mathrm{D}^4 + 8\mathrm{D}^2 - 9$ is $p(s) = s^4 + 8s^2 - 9$. Therefore its Green function $g(t)$ is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s^4 + 8s^2 - 9}\right](t)\,.$$

Because $p(s)$ factors as $p(s) = (s^2-1)(s^2+9)$ we have the partial fraction identity

$$\frac{1}{s^4 + 8s^2 - 9} = \frac{1}{(s^2-1)(s^2+9)} = \frac{\frac{1}{10}}{s^2-1} + \frac{-\frac{1}{10}}{s^2+9}\,.$$

Because $s^2 - 1$ factors as $s^2 - 1 = (s-1)(s+1)$ we have the partial fraction identity

$$\frac{1}{s^2-1} = \frac{1}{(s-1)(s+1)} = \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}}{s+1}\,.$$

By combining the above partial fraction identities we obtain

$$\frac{1}{s^4 + 8s^2 - 9} = \tfrac{1}{20}\frac{1}{s-1} - \tfrac{1}{20}\frac{1}{s+1} - \tfrac{1}{10}\frac{1}{s^2+9}\,.$$

Referring to the table on the last page, item 1 with $a = 1$ and $n = 0$ and with $a = -1$ and $n = 0$ gives

$$\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) = e^t\,, \qquad \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = e^{-t}\,,$$

while item 3 with $a = 0$ and $b = 3$ gives

$$\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right](t) = \sin(3t)\,.$$

Therefore the Green function $g(t)$ is given by

$$\begin{aligned}
g(t) &= \mathcal{L}^{-1}\left[\frac{1}{s^4 + 8s^2 - 9}\right](t) \\
&= \tfrac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) - \tfrac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) - \tfrac{1}{30}\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right](t) \\
&= \tfrac{1}{20}e^t - \tfrac{1}{20}e^{-t} - \tfrac{1}{30}\sin(3t)\,.
\end{aligned}$$

(8) Transform the equation $u''' + t^2 u' - 3u = \sinh(2t)$ into a first-order system of ordinary differential equations.

Solution. Because the equation is third order, the first-order system must have dimension three. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \sinh(2t) + 3x_1 - t^2 x_2 \end{pmatrix}, \qquad \text{where} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}.$$

(9) Consider two interconnected tanks filled with brine (salt water). The first tank contains 100 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 3 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 5 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 5 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution. The rates work out so there will always be 100 liters of brine in the first tank and 50 liters in the second. Let $S_1(t)$ be the grams of salt in the first tank and $S_2(t)$ be the grams of salt in the second tank. These are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 2\cdot 3 + \frac{S_2}{50}\,2 - \frac{S_1}{100}\,5\,, \qquad & S_1(0) &= 5\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{100}\,5 - \frac{S_2}{50}\,2 - \frac{S_2}{50}\,3\,, \qquad & S_2(0) &= 20\,. \end{aligned}$$

You could leave the answer in the form given above. However, it can be simplified to

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 6 + \frac{S_2}{25} - \frac{S_1}{20}\,, \qquad & S_1(0) &= 5\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{20} - \frac{S_2}{10}\,, \qquad & S_2(0) &= 20\,. \end{aligned}$$

(10) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices

(a) $\mathbf{A}^T$,
(b) $\overline{\mathbf{A}}$,
(c) $\mathbf{A}^*$,
(d) $5\mathbf{A} - \mathbf{B}$,
(e) $\mathbf{AB}$,
(f) $\mathbf{B}^{-1}$.

Solution (a). The transpose of $\mathbf{A}$ is

$$\mathbf{A}^T = \begin{pmatrix} -i2 & 2+i \\ 1+i & -4 \end{pmatrix}.$$

Solution (b). The conjugate of $\mathbf{A}$ is

$$\overline{\mathbf{A}} = \begin{pmatrix} i2 & 1-i \\ 2-i & -4 \end{pmatrix} .$$

Solution (c). The Hermitian transpose of $\mathbf{A}$ is

$$\mathbf{A}^* = \begin{pmatrix} i2 & 2-i \\ 1-i & -4 \end{pmatrix} .$$

Solution (d). The difference of $5\mathbf{A}$ and $\mathbf{B}$ is given by

$$5\mathbf{A} - \mathbf{B} = \begin{pmatrix} -i10 & 5+i5 \\ 10+i5 & -20 \end{pmatrix} - \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} -7-i10 & -1+i5 \\ 2+i5 & -27 \end{pmatrix} .$$

Solution (e). The product of $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -i2\cdot 7 + (1+i)\cdot 8 & -i2\cdot 6 + (1+i)\cdot 7 \\ (2+i)\cdot 7 - 4\cdot 8 & (2+i)\cdot 6 - 4\cdot 7 \end{pmatrix} \\ &= \begin{pmatrix} 8-i6 & 7-i5 \\ -18+i7 & -16+i6 \end{pmatrix} . \end{aligned}$$

Solution (f). Observe that it is clear that $\mathbf{B}$ has an inverse because

$$\det(\mathbf{B}) = \det\begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = 7\cdot 7 - 6\cdot 8 = 49 - 48 = 1 .$$

Then the inverse of $\mathbf{B}$ is given by

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} .$$

(11) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) Give a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b).

(d) For the system found in part (b), solve the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x} , \qquad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Solution (a).

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} = 3t^4 + 9 - 2t^4 = t^4 + 9\,.$$

Solution (b). Let $\mathbf{\Psi}(t) = \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}$. Because $\dfrac{\mathrm{d}\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t)$, we have

$$\begin{aligned}\mathbf{A}(t) &= \frac{\mathrm{d}\mathbf{\Psi}(t)}{\mathrm{d}t}\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix}\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}^{-1} \\ &= \frac{1}{t^4+9}\begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix}\begin{pmatrix} 3 & -t^2 \\ -2t^2 & t^4+3 \end{pmatrix} = \frac{1}{t^4+9}\begin{pmatrix} 8t^3 & 6t-2t^5 \\ 12t & -4t^3 \end{pmatrix}.\end{aligned}$$

Solution (c). Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a fundamental matrix for the system found in part (b) is simply given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}.$$

Solution (d). Because a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b) was given in part (c), the solution of the initial-value problem is

$$\begin{aligned}\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(1)^{-1}\mathbf{x}(1) &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\frac{1}{10}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3t^4+9-2t^2 \\ 6t^2-6 \end{pmatrix}.\end{aligned}$$

Alternative Solution (d). Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix} + c_2\begin{pmatrix} t^2 \\ 3 \end{pmatrix}.$$

The initial condition then implies that

$$\mathbf{x}(1) = c_1\begin{pmatrix} 4 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4c_1 + c_2 \\ 2c_1 + 3c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

from which we see that $c_1 = \frac{3}{10}$ and $c_2 = -\frac{1}{5}$. The solution of the initial-value problem is thereby

$$\mathbf{x}(t) = \tfrac{3}{10}\begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix} - \tfrac{1}{5}\begin{pmatrix} t^2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10}t^4 - \frac{1}{5}t^2 + \frac{9}{10} \\ \frac{3}{5}t^2 - \frac{3}{5} \end{pmatrix}.$$

(12) Compute $e^{t\mathbf{A}}$ for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

Solution (a). The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z-1)^2 - 4\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 1 ± 2. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\cosh(2t)\mathbf{I} + \frac{\sinh(2t)}{2}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[\cosh(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(2t)}{2} \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cosh(2t) & 2\sinh(2t) \\ \frac{1}{2}\sinh(2t) & \cosh(2t) \end{pmatrix} . \end{aligned}$$

Natural Fundamental Set Method Solution (a). The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z+1)(z-3)\,.$$

The associated second-order general initial-value problem is

$$y'' - 2y' - 3y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

This has the general solution $y(t) = c_1 e^{3t} + c_2 e^{-t}$. Because $y'(t) = 3c_1 e^{3t} - c_2 e^{-t}$, the general initial conditions yield

$$y_0 = y(0) = c_1 + c_2\,, \qquad y_1 = y'(0) = 3c_1 - c_2\,.$$

This system can be solved to obtain

$$c_1 = \frac{y_0 + y_1}{4}\,, \qquad c_2 = \frac{3y_0 - y_1}{4}\,.$$

The solution of the general initial-value problem is thereby

$$y(t) = \frac{y_0 + y_1}{4}\, e^{3t} + \frac{3y_0 - y_1}{4}\, e^{-t} = \frac{e^{3t} + 3e^{-t}}{4}\, y_0 + \frac{e^{3t} - e^{-t}}{4}\, y_1\,.$$

Therefore the associated natural fundamental set of solutions is

$$N_0(t) = \frac{e^{3t} + 3e^{-t}}{4}\,, \qquad N_1(t) = \frac{e^{3t} - e^{-t}}{4}\,,$$

whereby

$$\begin{aligned} e^{t\mathbf{A}} = N_0(t)\mathbf{I} + N_1(t)\mathbf{A} &= \frac{e^{3t} + 3e^{-t}}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{3t} - e^{-t}}{4} \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2e^{3t} + 2e^{-t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix} . \end{aligned}$$

Eigen Method Solution (a). The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z+1)(z-3)\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -1 and 3. Because

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix},$$

we can read off that $\mathbf{A}$ has the eigenpairs

$$\left(-1\,, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right), \qquad \left(3\,, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

Set

$$\mathbf{V} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Because $\det(\mathbf{V}) = 4$, we see that

$$\begin{aligned}
e^{t\mathbf{A}} &= \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & -2e^{-t} \\ e^{3t} & 2e^{3t} \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 2e^{-t} + 2e^{3t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{-t} + 2e^{3t} \end{pmatrix}.
\end{aligned}$$

Solution (b). The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z-4)^2\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is 4, a double root. Hence,

$$\begin{aligned}
e^{t\mathbf{A}} = e^{4t}\Big[\mathbf{I} + t\,(\mathbf{A} - 4\mathbf{I})\Big] &= e^{4t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}\right] \\
&= e^{4t}\begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix}.
\end{aligned}$$

Natural Fundamental Set Method Solution (b). The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z-4)^4\,.$$

The associated second-order general initial-value problem is

$$y'' - 8y' + 16y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,.$$

This has the general solution $y(t) = c_1e^{4t} + c_2te^{4t}$. Because $y'(t) = 4c_1e^{3t} + 4c_2te^{-t} + c_2e^{4t}$, the general initial conditions yield

$$y_0 = y(0) = c_1\,, \qquad y_1 = y'(0) = 4c_1 + c_2\,.$$

This system can be solved to obtain $c_1 = y_0$ and $c_2 = y_1 - 4y_0$. The solution of the general initial-value problem is thereby

$$y(t) = y_0\,e^{4t} + (y_1 - 4y_0)t\,e^{4t} = (1-4t)e^{4t}\,y_0 + t\,e^{4t}\,y_1\,.$$

Therefore the associated natural fundamental set of solutions is

$$N_0(t) = (1-4t)e^{4t}\,, \qquad N_1(t) = t\,e^{4t}\,,$$

whereby

$$\begin{aligned} e^{t\mathbf{A}} &= N_0(t)\mathbf{I} + N_1(t)\mathbf{A} = (1-4t)e^{4t}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + t\,e^{4t}\begin{pmatrix}6 & 4\\ -1 & 2\end{pmatrix} \\ &= e^{4t}\begin{pmatrix}1+2t & 4t\\ -t & 1-2t\end{pmatrix}\,. \end{aligned}$$

(13) Solve each of the following initial-value problems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}2 & 2\\ 5 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\,, \qquad \begin{pmatrix}x(0)\\ y(0)\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}\,.$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1 & 1\\ -4 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\,, \qquad \begin{pmatrix}x(0)\\ y(0)\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}\,.$

Solution (a). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}2 & 2\\ 5 & -1\end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - z - 12 = (z+3)(z-4)\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -3 and 4. These have the form $\frac{1}{2} \pm \frac{7}{2}$, whereby

$$\begin{aligned} e^{t\mathbf{A}} &= e^{\frac{1}{2}t}\left[\cosh\left(\tfrac{7}{2}t\right)\mathbf{I} + \frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}}\left(\mathbf{A} - \tfrac{1}{2}\mathbf{I}\right)\right] \\ &= e^{\frac{1}{2}t}\left[\cosh\left(\tfrac{7}{2}t\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}}\begin{pmatrix}\frac{3}{2} & 2\\ 5 & -\frac{3}{2}\end{pmatrix}\right] \\ &= e^{\frac{1}{2}t}\begin{pmatrix}\cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right)\\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right)\end{pmatrix}\,. \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\begin{aligned} \begin{pmatrix}x(t)\\ y(t)\end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix}x(0)\\ y(0)\end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix}1\\ -1\end{pmatrix} \\ &= e^{\frac{1}{2}t}\begin{pmatrix}\cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right)\\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right)\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} \\ &= e^{\frac{1}{2}t}\begin{pmatrix}\cosh\left(\frac{7}{2}t\right) - \frac{1}{7}\sinh\left(\frac{7}{2}t\right)\\ -\cosh\left(\frac{7}{2}t\right) + \frac{13}{7}\sinh\left(\frac{7}{2}t\right)\end{pmatrix}\,. \end{aligned}$$

Solution (b). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}1 & 1\\ -4 & 1\end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 5 = (z-1)^2 + 4\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm i2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[\cos(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2} \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix} . \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^t \begin{pmatrix} \cos(2t) + \frac{1}{2}\sin(2t) \\ -2\sin(2t) + \cos(2t) \end{pmatrix} . \end{aligned}$$

Remark. We could have used other methods to compute $e^{t\mathbf{A}}$ in each part of the above problem. Alternatively, we could have constructed a fundamental matrix $\mathbf{\Psi}(t)$ and then determined $\mathbf{c}$ so that $\mathbf{\Psi}(t)\mathbf{c}$ satisfies the initial conditions.

(14) Find a general solution for each of the following systems.

(a) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(c) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solution (a). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 1 = (z-1)^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is 1, a double root. Hence,

$$\begin{aligned} e^{t\mathbf{A}} = e^t \big[\mathbf{I} + t\,(\mathbf{A} - \mathbf{I}) \big] &= e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} . \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^t \begin{pmatrix} 1+2t \\ t \end{pmatrix} + c_2 e^t \begin{pmatrix} -4t \\ 1-2t \end{pmatrix} . \end{aligned}$$

Solution (b). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $\pm i4$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= \left[\cos(4t)\mathbf{I} + \frac{\sin(4t)}{4}\mathbf{A}\right] = \left[\cos(4t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(4t)}{4}\begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\right] \\ &= \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ \sin(4t) \end{pmatrix} + c_2\begin{pmatrix} -\frac{5}{4}\sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

Eigen Method Solution (b). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $\pm i4$. Because

$$\mathbf{A} - i4\mathbf{I} = \begin{pmatrix} 2 - i4 & -5 \\ 4 & -2 - i4 \end{pmatrix}, \qquad \mathbf{A} + i4\mathbf{I} = \begin{pmatrix} 2 + i4 & -5 \\ 4 & -2 + i4 \end{pmatrix},$$

we can read off that $\mathbf{A}$ has the eigenpairs

$$\left(i4\,, \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix}\right), \qquad \left(-i4\,, \begin{pmatrix} 1 - i2 \\ 2 \end{pmatrix}\right).$$

Therefore the system has the complex-valued solution

$$\begin{aligned} e^{i4t}\begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} &= \big(\cos(4t) + i\sin(4t)\big)\begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \cos(4t) - 2\sin(4t) + i2\cos(4t) + i\sin(4t) \\ 2\cos(4t) + i2\sin(4t) \end{pmatrix}. \end{aligned}$$

By taking real and imaginary parts, we obtain the two real solutions

$$\begin{aligned} \mathbf{x}_1(t) &= \operatorname{Re}\left(e^{i4t}\begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} \cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix}, \\ \mathbf{x}_2(t) &= \operatorname{Im}\left(e^{i4t}\begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\begin{pmatrix} \cos(4t) - 2\sin(4t) \\ 2\cos(4t) \end{pmatrix} + c_2\begin{pmatrix} 2\cos(4t) + \sin(4t) \\ 2\sin(4t) \end{pmatrix}.$$

Solution (c). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z-3)^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $3 \pm i4$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t}\left[\cos(4t)\mathbf{I} + \frac{\sin(4t)}{4}(\mathbf{A} - 3\mathbf{I})\right] \\ &= e^{3t}\left[\cos(4t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(4t)}{4}\begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix}\right] \\ &= e^{3t}\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= e^{3t}\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^{3t}\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ -\frac{5}{4}\sin(4t) \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} \sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}. \end{aligned}$$

Eigen Method Solution (c). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z-3)^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $3 \pm i4$. Because

$$\mathbf{A} - (3+i4)\mathbf{I} = \begin{pmatrix} 2 - i4 & 4 \\ -5 & -2 - i4 \end{pmatrix}, \quad \mathbf{A} - (3-i4)\mathbf{I} = \begin{pmatrix} 2 + i4 & 4 \\ -5 & -2 + i4 \end{pmatrix},$$

we can read off that $\mathbf{A}$ has the eigenpairs

$$\left(3 + i4\,, \begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix}\right), \qquad \left(3 - i4\,, \begin{pmatrix} -2 \\ 1 + i2 \end{pmatrix}\right).$$

Therefore the system has the complex-valued solution

$$\begin{aligned} e^{(3+i4)t}\begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} &= e^{3t}\big(\cos(4t) + i\sin(4t)\big)\begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix} \\ &= e^{3t}\begin{pmatrix} -2\cos(4t) - i2\sin(4t) \\ \cos(4t) + 2\sin(4t) + i\sin(4t) - i2\cos(4t) \end{pmatrix}. \end{aligned}$$

By taking real and imaginary parts, we obtain the two real solutions

$$\begin{aligned} \mathbf{x}_1(t) &= \operatorname{Re}\left(e^{(3+i4)t}\begin{pmatrix} -2 \\ 1 - i2 \end{pmatrix}\right) = e^{3t}\begin{pmatrix} -2\cos(4t) \\ \cos(4t) + 2\sin(4t) \end{pmatrix}, \\ \mathbf{x}_2(t) &= \operatorname{Im}\left(e^{(3+i4)t}\begin{pmatrix} -2 \\ 1 + i2 \end{pmatrix}\right) = e^{3t}\begin{pmatrix} -2\sin(4t) \\ \sin(4t) - 2\cos(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t}\begin{pmatrix} -2\cos(4t) \\ \cos(4t) + 2\sin(4t) \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} -2\sin(4t) \\ \sin(4t) - 2\cos(4t) \end{pmatrix}.$$

A Short Table of Laplace Transforms

$\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$	for $s > a$.
$\mathcal{L}[e^{at}\cos(bt)](s) = \dfrac{s-a}{(s-a)^2+b^2}$	for $s > a$.
$\mathcal{L}[e^{at}\sin(bt)](s) = \dfrac{b}{(s-a)^2+b^2}$	for $s > a$.
$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[e^{at} j(t)](s) = J(s-a)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s)$	where $J(s) = \mathcal{L}[j(t)](s)$ and u is the unit step function.