Quiz 6 Solutions, Math 246, Professor David Levermore
Thursday, 16 March 2017

(1) [5] Find a particular solution of $v'' - v = 4e^t$.

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2 - 1$, which has roots ± 1. Its forcing has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = 1$, and multiplicity $m = 1$. Let $\mathrm{L} = \mathrm{D}^2 - 1$.

Zero Degree Formula. Because $d = 0$, $\mu + i\nu = 1$, and $m = 1$, we may use the zero degree formula with $m = 1$. Because $p'(z) = 2z$, this formula is

$$\mathrm{L}\left(\frac{t\,e^t}{p'(1)}\right) = \mathrm{L}\left(\frac{t\,e^t}{2}\right) = e^t\,.$$

Therefore a particular solution of $\mathrm{L}(v) = 4e^t$ is $v_P(t) = 2t\,e^t$.

Remark. Had you forgotten the zero degree formula then you could have derived it by Key Identity Evaluations as in the following solution.

Key Identity Evaluations. Because $d = 0$, $\mu + i\nu = 1$, and $m = 1$, we need to evaluate the first derivative of the Key Identity at $z = 1$. Because $p(z) = z^2 + 1$, the Key Identity and its first derivative with respect to z are

$$\mathrm{L}(e^{zt}) = p(z)e^{zt} = (z^2 - 1)e^{zt}\,,$$
$$\mathrm{L}(t\,e^{zt}) = p(z)t\,e^{zt} + p'(z)e^{zt} = (z^2 - 1)t\,e^{zt} + 2ze^{zt}\,.$$

By evaluating the first derivative of the Key Identity at $z = 1$ we obtain

$$\mathrm{L}(t\,e^t) = (1^2 - 1)t\,e^t + 2 \cdot 1e^t = 2e^t\,.$$

Therefore a particular solution of $\mathrm{L}(v) = 4e^t$ is $v_P(t) = 2t\,e^t$.

Undetermined Coefficients. Because $d = 0$, $\mu + i\nu = 1$ and $m = 1$, there is a particular solution of $\mathrm{L}(v) = 4e^t$ in the form

$$v_P(t) = At\,e^t\,.$$

Because

$$v_P'(t) = At\,e^t + Ae^t\,, \qquad v_P''(t) = At\,e^t + 2Ae^t\,,$$

we find that

$$v_P''(t) - v_P(t) = \big(At\,e^t + 2Ae^t\big) - At\,e^t = 2Ae^t\,.$$

By setting $2Ae^t = 4e^t$ we see that $2A = 4$, whereby $A = 2$. Therefore a particular solution of $\mathrm{L}(v) = 4e^t$ is $v_P(t) = 2t\,e^t$.

(2) [5] Find a particular solution of $w'' + w = 6\cos(2t)$.

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is $p(z) = z^2 + 1$, which has roots $\pm i$. Its forcing has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = i2$, and multiplicity $m = 0$. Let $\mathrm{L} = \mathrm{D}^2 + 1$.

Zero Degree Formula. Because $d = 0$, $\mu + i\nu = i2$, and $m = 0$, we may use the zero degree formula with $m = 0$. Because $p(z) = z^2 + 1$, this formula is

$$\mathrm{L}\left(\frac{e^{i2t}}{p(i2)}\right) = \mathrm{L}\left(\frac{e^{i2t}}{-2^2 + 1}\right) = \mathrm{L}\big(-\tfrac{1}{3}e^{i2t}\big) = e^{i2t}\,.$$

Because $6\cos(2t) = 6\,\mathrm{Re}(e^{i2t})$, we see that a particular solution of $\mathrm{L}(w) = 6\cos(2t)$ is

$$w_P(t) = 6\,\mathrm{Re}\big(-\tfrac{1}{3}e^{i2t}\big) = -2\,\mathrm{Re}\big(e^{i2t}\big) = -2\cos(2t)\,.$$

Remark. Had you forgotten the zero degree formula then you could have derived it by Key Identity Evaluations as in the following solution.

Key Identity Evaluations. Because $d = 0$, $\mu + i\nu = i2$, and $m = 0$, we just need to evaluate the Key Identity at $z = i2$. Because $p(z) = z^2 + 1$, this identity is

$$\mathrm{L}(e^{zt}) = p(z)e^{zt} = (z^2 + 1)e^{zt}\,.$$

By evaluating this at $z = i2$ we obtain

$$\mathrm{L}(e^{i2t}) = (-2^2 + 1)e^{t} = -3e^{i2t}\,.$$

Because $6\cos(2t) = 6\,\mathrm{Re}(e^{i2t})$, we see that a particular solution of $\mathrm{L}(w) = 6\cos(2t)$ is

$$w_P(t) = -2\,\mathrm{Re}\big(e^{i2t}\big) = -2\cos(2t)\,.$$

Undetermined Coefficients. Because $d = 0$, $\mu + i\nu = i2$ and $m = 0$, there is a particular solution of $\mathrm{L}(w) = 6\cos(2t)$ in the form

$$w_P(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$w_P'(t) = -2A\sin(2t) + 2B\cos(2t)\,, \qquad w_P''(t) = -4A\cos(2t) - 4B\sin(2t)\,,$$

we find that

$$\begin{aligned} w_P''(t) + w_P(t) &= \big(-4A\cos(2t) - 4B\sin(2t)\big) + \big(A\cos(2t) + B\sin(2t)\big) \\ &= -3A\cos(2t) - 3B\sin(2t)\,. \end{aligned}$$

By setting $-3A\cos(2t) - 3B\sin(2t) = 6\cos(2t)$, we see that $-3A = 6$ and $-3B = 0$, whereby $A = -2$ and $B = 0$. Therefore a particular solution of $\mathrm{L}(w) = 6\cos(2t)$ is $w_P(t) = -2\cos(2t)$.

Remark. Neither of these problems had a forcing with positive degree or composite characteristic form. Be prepared for either of these cases on the exam. They are included among the sample problems.