On the Delay and Throughput of Digital and Analog Network Coding for Wireless Broadcast

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Abstract—We address the problem of exchanging broadcast packets among multiple wireless terminals through a single relay node. The objective is to evaluate the delay and throughput gains of network coding over plain routing. We compare digital network coding at the packet level with analog network coding based on scheduled or random access of terminal transmissions that are forwarded by the relay node. For error-free channels, the performance gain of both types of network coding scales with the number of terminals, if they can overhear each other’s transmissions. For channels with noise or packet erasures, we formulate network coding as a multiuser communication problem. The multi-dimensional performance measures involve the packet delay, the throughput rate and the probability of decoding error or decoding failure that are optimized either by plain routing, digital or analog network coding depending on the number of terminals and channel properties. Our results open up new questions regarding the use of wireless network coding and illustrate the delay, throughput and reliability trade-offs.

I. INTRODUCTION

Network coding maximizes the achievable throughput rates for single-source multicast communication in wired networks [1]. The extension to wireless networks involves the joint design of network codes with medium access control under the additional properties of omnidirectional transmissions, half-duplex operation and interference effects [2]-[3]. To gain insights on the open problem of multi-source network coding, we consider a simple wireless network with multiple sources exchanging broadcast packets through a single relay node.

Since signals are received as their superposition in wireless access, the simple procedure of forwarding signals (without decoding them first) enables analog network coding in the physical layer [4]-[5]. Much of the work on multi-source network coding in wireless networks has focused on a canonical three-node network in which two terminals want to send traffic to each other with the assistance of a third relay node. Analog network coding has been formulated for the two-way and multicast-relay-channels [6]-[7] as an extension to the amplify-and-forward mechanism for the two-way relay channels [8].

In this paper, we consider a natural generalization of such models in which $K \geq 2$ nodes all want to broadcast traffic to each other, again with the aid of a single relay. This allows us to study how the performance of the classical approach of digital network coding at the packet level and analog network coding at the signal level scales with the size of the system. For error-free channels, we evaluate the packet delay and quantify the effects of packet overhearing between terminal pairs. Analog network coding can minimize the decoding delay compared to digital network coding by carefully scheduling the transmissions of terminals. A distributed solution based on random access of terminals generates random network codes [9] at the signal level and is shown to approach the throughput of scheduled access, as the network size grows.

For noisy channels, we analyze network coding as a multiuser communication problem under a finite delay constraint. The error performance of digital network coding has been analyzed in [10] at the packet level. Instead, we propose a general multiuser communication framework that captures both digital and analog network coding based on the noisy observations of the transmissions at the packet or signal level. First, we evaluate the end-to-end probability of decoding error with finite delay. Then, we can abstract each source-destination pair as a single channel and impose channel coding (over a very long period of time) to optimize the achievable rates for reliable communication. On the other hand, deep channel fading can be modeled by erasure channels. The performance objective is then to minimize the probability of decoding failure due to packet losses. Different schemes of plain routing, digital and analog network coding may outperform each other (depending on the number of terminals and channel noise or packet erasure properties) in terms of the packet delay, the achievable rate and the decoding error or decoding failure.

The key results are summarized as follows: (i) For error-free channels, analog network coding outperforms digital network coding in terms of delay and throughput; (ii) The performance improvement of either type of network coding over plain routing scales with the network size, only if sources can overhear each other’s transmissions; (iii) For channels with noise, network coding can be formulated as a multiuser communication problem with strong trade-offs involving the multiple performance criteria of delay, throughput and reliability (that can be measured by the probability of decoding error); (iv) Packet losses due to channel fading can be modeled by erasure channels and the resulting probability of decoding failure strongly depends on the number of terminals as well as on which routing or network coding scheme is used.

The paper is organized as follows. Section II presents the network and communication model. This is followed in Section III by the delay and throughput performance comparison of (analog and digital) network coding and plain routing for noiseless channels with possible packet overhearing. We introduce in Section IV a general multiuser communication...
framework for network coding over channels with additive noise or with packet erasures. Finally, we summarize the results and present thoughts for future work in Section V.

II. NETWORK AND COMMUNICATION MODEL

We consider $K \geq 2$ terminals uniformly located around a relay $R$, as shown in Figure 1. We assume broadcast communication where each terminal has packets to be delivered to all of the other terminals. We consider omnidirectional transmissions such that a relay transmission reaches all terminals, a terminal transmission reaches relay $R$ and any terminal $i$ can overhear transmissions of $n$ neighbors on each side, collectively denoted as set $N_i$ ($n = 0$ corresponds to no overhearing). Half-duplex communication (such that a node cannot simultaneously transmit and receive packets) is assumed in a synchronous slotted system (i.e., each transmission takes one time slot).

![Fig. 1. Star topology with noise or with packet erasures. Finally, we summarize the framework for network coding over channels with additive noise or with packet erasures. Finally, we summarize the results and present thoughts for future work in Section V.](image)

We consider error-free channels in Section III and introduce additive channel noise and packet erasures in Section IV. We assume the following two-phase operation (separated in time)\(^1\):

(a) Phase 1: Terminals transmit packets to relay $R$ (with possible overhearing among each other), and

(b) Phase 2: Relay $R$ broadcasts packets/signals back to terminals.

We distinguish three different transmission schemes:

(a) Plain Routing: Terminals transmit packets over separate time slots in Phase 1 and relay $R$ transmits one plain (not network-coded) packet at a time in Phase 2.

(b) Digital Network Coding [1]: Relay $R$ decodes and re-encodes the packets incoming in Phase 1 before the subsequent transmission in Phase 2. We assume linear network coding [11] in finite field $F_q$ with the coding size $q$.

(c) Analog Network Coding [4-5]: Relay $R$ receives a linear combination of signals transmitted in Phase 1 and forwards them in Phase 2 without decoding them first\(^2\).

Any coding configuration is kept in the packet header and transmitted with the data packet. We assume that the packet overhead is negligible compared to the data packet length.

III. PLAIN ROUTING VS. (ANALOG OR DIGITAL) NETWORK CODING FOR NOISELESS CHANNELS

The objective is to evaluate the packet delay, namely the number of time slots $T(K)$ necessary to exchange one packet per terminal. Define $\mathbf{x} \in \mathcal{A}^K$ as the information packets

\(^1\)This two-phase operation is not necessarily throughput-optimal, since we do not allow packets to be relayed over the neighboring terminals.

\(^2\)We ignore the limits on receiver’s dynamic range and quantization effects.

\(^3\)We use the lower-case and upper-case notation for packets and signals.

\[ \mathbf{x} = \mathcal{M}(\mathbf{x}) \]  

for the one-to-one modulation $\mathcal{M}$. Terminal $i$ demands packets $\mathbf{x}^i = \mathbf{x} - \{x_i\}$. Let $\mathbf{y}^{i,1}$ and $\mathbf{y}^{i,2}$ denote the signal vectors received by terminal $i$ in Phase 1 and Phase 2, respectively. Let $\mathbf{y}^R$ and $\mathbf{x}^R$ denote the signal vectors received and transmitted by relay $R$ in Phase 1 and Phase 2, respectively, such that

\[ \mathbf{y}^R = \mathcal{F}^R(\mathbf{y}_R), \]

where the relay operation $\mathcal{F}^R$ depends on the transmission schemes. We consider error-free transmissions such that the only constraint is the finite packet delay under interference effects (i.e., we do not consider channel coding over a long period of time to optimize the achievable rates).

![Fig. 2. Channel model in the two-phase operation.](image)

Figure 2 shows the channels described for any terminal $i$ as

\[ \mathbf{y}^{i,1} = H^{i,1} \mathbf{x}, \]

\[ \mathbf{y}^R = H^R \mathbf{x}, \]

\[ \mathbf{y}^{i,2} = H^{R,i} \mathbf{x}^R, \]

where $H^{i,1}$ is the channel transition matrix from all terminals to terminal $i$ in Phase 1, $H^R$ is the channel transition matrix from all terminals to relay $R$ in Phase 1 and $H^{R,i}$ is the channel transition matrix from relay $R$ to terminal $i$ in Phase 2.

Define $h_{i,j}^{t,m} \neq 0$ as the channel gain from node $i$ to node $j$ at time slot $t$ of Phase $m = 1, 2$. Assume that the symbols in any given packet are subject to the same channel gains. Since terminal $i$ cannot transmit and receive packets simultaneously, the non-zero entries of the channel transition matrices are

\[ H_{i,j}^{t,1} = \tilde{G}_{i,j} (1 - \tilde{G}_{i,j}) h_{i,j}^{t,1}, \]

\[ H_{i,j}^{R,i} = \tilde{h}_{R,i} h_{i,j}^{m} \]

where $\tilde{G}$ is the binary scheduling matrix for Phase 1 such that

\[ \tilde{G}_{i,j} = \begin{cases} 
1, & \text{if terminal } j \text{ transmits in time slot } t, \\
0, & \text{otherwise}, 
\end{cases} \]

which depends on the transmission scheme. Define $\tilde{M}^i$ as the $m \times (K - 1)$ end-to-end channel transition matrix for terminal $i$ (where $m$ is the total number of rows in $\mathbf{y}^{i,1}$ and $\mathbf{y}^{i,2}$) from the finite field under plain routing or digital network coding, or from the infinite field ($\mathbb{R}$ or $\mathbb{C}$) under analog network coding.

Consider randomly varying channel gains such that the rank deficiency of the channel transition matrix $\tilde{M}^i$ for any terminal $i$ depends only on the transmission decisions of terminals (and not on the channel gains). Assume full channel state

\[ M^i \]

\[ M^i \]

\[ M^i \]
information at each node. If terminal $i$ receives over two phases the packets
\[ y^i = M^i x^i \]  
for routing or digital network coding, or receives the signals
\[ Y^i = M^i X^i \]
for analog network coding, terminal $i$ can decode $x^i$, only if
\[ \text{rank}(M^i) = K - 1. \]

A. Plain Routing

Replicate-and-forward-based plain routing is described by
\[ \tilde{G} = I_K, \]
\[ X^R = X, \]
where $I_K$ is the $K \times K$ identity matrix.

**Theorem 1:** The minimum delay achievable by plain routing is given by
\[ T(K) = K(1 + 1_{\{K>2n+1\}}), \quad K \geq 2, \]
where $1_{\{\cdot\}}$ is the indicator function.

**Proof:** Plain routing requires $K$ packets to be separately delivered to relay $R$ in Phase 1 and then forwarded to terminals over additional $K$ transmissions in Phase 2. If $2n \geq K - 1$, Phase 1 of $K$ time slots is sufficient to deliver all packets to $K - 1$ destinations by overhearing. Therefore, the minimum delay achievable by plain routing is given by Eq. (15).  

B. Digital Network Coding

Relay $R$ decodes the packet of each terminal received in Phase 1, re-encodes the packets and transmits them in Phase 2. Digital network coding is described by
\[ \tilde{G} = I_K, \]
\[ X^R = M(G \cdot X), \]
where $G$ is the linear coding matrix ($G_{i,j}$ is the coding coefficient for packet $x_j$ of terminal $j$ in the $i$th coded packet).

**Theorem 2:** The minimum delay achievable by digital network coding is given by
\[ T(K) = \max (K, 2K - 1 - 2n), \quad K \geq 2. \]  

**Proof:** Phase 1 clearly requires $K$ transmissions. In Phase 2, each transmission represents a linear equation in terms of the $K$ packets. If $n = 0$, each user requires $K - 1$ independent equations to be able to solve for the $K - 1$ needed packets. This requires that $G$ has at least $K$ rows. For packets of bits, the following is an example of $G$ that meets this bound:
\[ G = [I_{K-1}, 1_{K-1}], \]
where $1_{K-1}$ is the $(K - 1)$-dimensional column vector of all 1's. This coding matrix $G$ has the minimum number of $K - 1$ rows needed to have the rank of $K - 1$. If we allow terminals to overhear transmissions over $n$ neighbor hops, $2n$ degrees of freedom are delivered to each terminal in Phase 1. Here, a degree of freedom received at node $i$ represents the change in the rank of $M^i$. Then, at least $K - 1 - 2n$ transmissions are necessary in Phase 2 to deliver $K - 1$ linearly independent coded transmissions of $x^i = x - \{x_i\}$ to each terminal $i$.

Next, we show that $K - 1 - 2n$ coded transmissions are sufficient to form the decoding matrix of rank $K$ at any terminal. Define $g_i$ as the $i$th column of $G$, and define matrix $G_i$ by removing the adjacent columns $g_i$ and $g_j$, $j \in N_i$, from $G$. If $G_i$ has full column rank, terminal $i$ can decode $K - 1 - 2n$ packets missing from Phase 1. The optimal coding matrix $G$ is not unique. To construct $G$, we define the $r_i \times s_i$ matrix
\[ U_i = [I_{r_i}, \ldots, I_{r_i}, V_i], \]
where $r_{i+1} = \text{mod}(r_i, s_{i+1})$ and $s_{i+1} = \text{mod}(s_i, r_i)$ with the initial conditions of $r_0 = K - 1 - 2n$ and $s_0 = K$, and the $r_i \times s_{i+1}$ matrix $V_i$ is defined as
\[ V_i = [I_{s_{i+1}}, \ldots, I_{s_{i+1}}, U_i^T]_i^T, \]
where $\{\cdot\}^T$ is the matrix transpose. Then, an example of the $(K - 1 - 2n) \times K$ optimal coding matrix is $G = U_0$. This coding matrix $G$ evaluated at $n = 0$ is equal to the coding matrix given by (19). Any $K - 1 - 2n$ adjacent columns of matrix $G$ are independent, i.e., the matrix $G_i$ is full-rank, and any terminal $i$ can decode $K - 1 - 2n$ missing packets. If $K - 1 - 2n \leq 0$, $K$ transmissions of Phase 1 are sufficient to decode all packets. As a result, the minimum delay achievable by digital network coding is given by Eq. (18).

C. Analog Network Coding

Terminals are scheduled to transmit packets in Phase 1 and the received signals are simply forwarded by relay $R$ in Phase 2 without coding. Analog network coding is described by
\[ \tilde{G} = G, \]
\[ X^R_t = \alpha_t Y^R_t, \]
for the scheduling matrix $G$ and the amplification factor $\alpha_t = \sqrt{P_R/\sum_{j=1}^K \|H^R_{ij}\|^2 P_j}$, where $P_j$ and $P_R$ are the transmitter powers of terminal $j = 1, \ldots, K$ and relay $R$.

**Theorem 3:** The minimum delay achievable by analog network coding is given by
\[ T(K) = \max (2 \lceil K/2 \rceil, 2(K - 1 - 2n)), \quad K \geq 2. \]

**Proof:** For $n = 0$, relay $R$ needs to deliver $K - 1$ degrees of freedom to Phase 2 to each terminal over at least $K - 1$ transmissions, i.e., $T(K) \geq 2(K - 1)$. If the scheduling matrix $G = G$ is chosen as (19), each terminal $i$ receives $K - 1$ linearly independent combinations of $K$ signals and can decode $K - 1$ missing signals by using own signal $M(x_i)$. The minimum achievable delay is $2(K - 1)$. For randomly varying channel gains, the rank of $M^i$ is $K - 1$ with probability one, if all terminals continuously transmit in Phase 1 without overhearing. At most $2n$ degrees of freedom can be delivered to each terminal by overhearing in Phase 1 and we need at least $K - 1 - 2n$ transmissions in Phase 2 such that $T(K) \geq 2(K - 1 - 2n)$. At least $K$ transmissions are needed to exchange $K$ packets and transmissions of Phase 1 are repeated in Phase 2 such that $T(K) \geq 2 \lceil K/2 \rceil$. The optimal scheduling matrix is not unique. Consider the $m \times K$ scheduling matrix $\tilde{G} = G$
\[ G = [U, V], \]
where \( m = \max\left(\left\lceil K/2 \right\rceil, K - 1 - 2n\right) \) and the matrices \( U \) and \( V \) are given by

\[
U_{i,j} = \begin{cases} 1, & \text{if } i = j \text{ or } \{i = K - s + 1, \ldots, K, \\ j = 1, \ldots, n+1\} \\ 0, & \text{otherwise}, \end{cases}
\]

\[
V = [I_r, \ldots, I_r, W^T]^T,
\]

where \( r = \text{mod}(K, m), s = \text{mod}(m, r) \), and

\[
W_{i,j} = \begin{cases} 1, & \text{if } i = K - s + 1, \ldots, K, j = r - n, \ldots, r \\ 0, & \text{otherwise}. \end{cases}
\]

D. Geometric Model for Packet Overhearing

Define \( d \) as the distance over which each terminal can overhear the packet transmission from another terminal. Define the distance between relay \( R \) and any terminal as \( r \). Assume that terminals can reach the relay independent of \( r \). The overhearing range \( n \) can be expressed in terms of \( K \) as

\[
n = \left\lceil \left( K/\pi \right) \sin^{-1}(d/2r) \right\rceil.
\]

For \( d = r \), we have \( n = \left\lceil \frac{\sqrt{2} K}{\pi} \right\rceil \). As \( K \) grows to infinity, \( T(K) \) approaches \( \frac{2K^2}{3} \) from (18) for digital network coding and \( \frac{4K^2}{3} \) from (24) for analog network coding with the respective throughput improvement of 20% and 50% over plain routing. From Eqs. (15), (18) and (24), the order of throughput improvement is illustrated in Figure 3, as \( K \) goes to infinity.

E. Analog Network Coding with Random Access

Instead of scheduling the terminal transmissions in Phase 1 (which requires large overhead), we can consider random access such that each terminal \( i \) transmits with probability \( p_i \) in Phase 1. This results in random signal arrivals analogous to random network coding [9] and allows terminals to overhear each other’s packets. Figure 4 shows the average throughput of analog network coding under random or scheduled access for \( n = 0 \) (no overhearing) or for \( n = 1 \) (one-hop overhearing). As \( K \to \infty \), the transmission probabilities \( p_i = 0.5, i = 1, \ldots, K \), become optimal and \( \lim_{K \to \infty} \frac{T(K)}{K} = 2 \) for finite \( n \).

IV. NETWORK CODING WITH MULTIUSER DETECTION

Assume channels with additive noise (denoted by \( N \)). Packet decoding is formulated as the detection over erroneous channels with non-zero decoding error under the finite delay constraint. Terminal \( i = 1, \ldots, K \) and relay \( R \) transmit with power \( P_j \) and \( P_R \), respectively. The channel model is given by

\[
Y_{i,1} = H_{i,1}^* X + N_{i,1}^*, \quad Y_R = H_R^* X + N_R^*, \quad Y_{i,2} = H_{R,i}^* X^R + N_{R,i}^*,
\]

for transmissions to terminal \( i \) and to relay \( R \) in Phase 1 and from relay \( R \) to terminal \( i \) in Phase 2, respectively. Define the aggregate set of observations at terminal \( i \) as

\[
Y^i = \begin{bmatrix} Y_{i,1}^* \\ Y_{i,2}^* \end{bmatrix} = \begin{bmatrix} H_{i,1}^* \\ H_{i,2}^* \end{bmatrix} X + \begin{bmatrix} N_{i,1}^* \\ N_{i,2}^* \end{bmatrix}.
\]

Let \( x_j \) denote the packet of terminal \( j \) decoded by terminal \( i \). Decoding is formulated according to the detection rule

\[
\hat{x}_i = F_i(Y_{i,1}^*, Y_{i,2}^*, x_i)
\]
based on the observations $Y^i$ and own packet $x_i$ at terminal $i$. The performance criterion is the probability of decoding error

$$p^R_{i,j} = P(S^i_j \neq x_j), \; j \neq i,$$

(37)

for any terminal pair $(i,j)$, and depends on the modulation scheme (e.g., BPSK, QPSK, etc.), multiuser detection method (e.g., decorrelator, successive interference cancellation (SIC), etc.) and coding or scheduling matrix $G$. Network coding is formulated as a multiuser communication problem:

Choose $G$, $F^R$, $\{F^i\}_{i=1}^K$, to minimize $C\left(\{p^R_{i,j}\}_{j \neq i}\right)$. (38)

where $C$ is any objective function (e.g., average or maximum) of the decoding error probabilities$^4$.

A. Digital Network Coding for Noisy Channels

Consider a decode-and-forward mechanism and formulate packet-level coding at relay $R$ as

$$X^R = F^R(Y^R) = M(G \cdot M^{-1}(F^R (Y^R))),$$

(39)

where $F^R(Y^R)$ is the overall relay operation at relay $R$, $F^R(Y^R)$ is the detection of $X$ by relay $R$ at the signal level using the signals $Y^R$, $G$ is the coding matrix. $M$ and $M^{-1}$ are the modulation and demodulation functions. Consider packets of bits. We model the channel between any transmitter-receiver pair as a binary symmetric channel with input packet $x$ and binary additive noise $n(x)$ such that

$$M^{-1}(f(hM(x) + N)) = x + n(x)$$

(40)

for any given (one-hop) channel gain $h$, channel noise $N$ and single-user signal detection rule $F$. We have $n(x) = n$ in (40) for the basic modulation schemes such as BPSK and QPSK. The input/output relationship at the packet level is given by

$$y^{i,1} = x + n^{i,1},$$

$$y^{i,2} = Gx + Gn^R + n^{R,i},$$

(41)

(42)

where $y^{i,1}$ is the output of the binary channel from the rest of terminals to terminal $i$ in Phase 1 and $y^{i,2}$ is the cascaded output of binary channels from terminals to relay $R$ in Phase 1 and from relay $R$ to terminal $i$ in Phase 2. For digital network coding, packet detection at any terminal $i$ is expressed as

$$\hat{x}^i = F_i(y^{i,1}, y^{i,2}, x_i).$$

(43)

B. Analog Network Coding for Noisy Channels

Amplify-and-forward operation at relay $R$ is expressed as

$$X^R = [F^R(Y^R)]_i = \alpha_i Y^R,$$

(44)

where the amplification factor for relay node $R$ is given by

$$\alpha_i = \sqrt{P_R/(\sum_{j=1}^K \|H^R_{i,j}\|^2 P_j + E(\|N^R\|^2))},$$

if the noise term $N^R$ is independent of signals $X$ and has zero mean. Terminal $i$ detects the signals

$$\hat{X}^i = F_i(Y^{i,1}, Y^{i,2}, M(x_i))$$

(45)

and demodulates the demanded packets $\hat{x}^i$ as

$$\hat{x}^i = M^{-1}(\hat{X}^i).$$

(46)

$^4$We can increase the sample space $Y^i$ and reduce $p^R_{i,j}$ at the expense of increasing the delay that may be also reflected in the objective function $C$.

C. Channels with Gaussian Noise

Assume that the channel noise is spatially and temporally independent, and has Gaussian distribution with zero mean and variance $\sigma^2$. Consider packets of bits and BPSK modulation$^5$

with $[M(x)]_i = 1$, if $x_i = 1$, or $[M(x)]_i = -1$, if $x_i = 0$, i.e., each node transmits with unit power. Assume coding/scheduling matrix $G$ given by (19) and consider noise variance $\sigma^2 = 0.1$, channel gains $h_{j,i}^{t,1} = 0.25$ for $i \neq j$, $h_{j,i}^{t,2} = 1$ and $h_{R,j}^{t} = 1$. Figure 5 shows the average probability of decoding error $p^R_{avg}$ is $\frac{1}{K(K-1)} \sum_{j=1}^K \sum_{i=1, i \neq j}^K p^R_{i,j}$ that is minimized by plain routing. Digital network coding achieves smaller values of $p^R_{avg}$ compared to analog network coding.

We can also abstract channels between source-destination pairs $(j,i)$ through a cascade of binary symmetric channels with the overall cross-over probability $p^R_{i,j}$, and impose channel coding for reliable communication over binary symmetric channels. The broadcast rate achievable by terminal $j$ is

$$R_j = \frac{K-1}{T(K)} \min_{i \neq j} \left(1 - H_2(p^R_{i,j})\right),$$

(47)

where $H_2(p) = -p \log_2 p - (1-p) \log_2 (1-p)$ is the binary entropy function. Figure 6 shows the average achievable rate $R^{avg} = \frac{1}{K} \sum_{j=1}^K R_j$ that is maximized by analog network coding with successive interference cancellation for small values of $K$. Digital network coding outperforms plain routing and maximizes $R^{avg}$, as $K$ increases.

D. Erasure Channels

Consider no additive noise but assume deep channel fading such that transmission of node $i$ fails at node $j$ with probability $\epsilon_{i,j}^{m}$ or successfully received by node $j$ with probability $1 - \epsilon_{i,j}^{m}$ at time slot $t$ of Phase $m = 1, 2$. The routing and coding/decoding operations continue without retransmitting the erased packets. Assume $\epsilon_{i,j}^{t,1} = 0.5$ for $i \neq j$, $\epsilon_{j,R}^{t,1} = 0.1$ and $\epsilon_{R,j}^{t,1} = 0.1$. Figure 7 depicts the average probability of decoding failure $p^f_{avg}$ is $\frac{1}{K(K-1)} \sum_{j=1}^K \sum_{i=1, i \neq j}^K p^f_{i,j}$, where $p^f_{i,j}$ is the probability that terminal $i$ fails to decode packet $x_j$ of terminal $j$ due to packet losses. Plain routing minimizes $p^f_{avg}$ whereas digital network coding achieves smaller values of $p^f_{avg}$ compared to analog network coding. As erasure probabilities for overhearing increase, the values of $p^f_{avg}$ achievable by digital and analog network coding approach each other.

We can abstract each source-destination pair $(j,i)$ as a binary erasure channel with erasure probability $p^e_{i,j}$. For reliable communication, we can impose channel coding over erasure channels. The broadcast rate achievable by terminal $j$ is

$$R_j = \frac{K-1}{T(K)} \min_{i \neq j} \left(1 - p^e_{i,j}\right).$$

(48)

Figure 8 shows the average achievable rate $R^{avg} = \frac{1}{K} \sum_{j=1}^K R_j$ that is maximized by analog network coding. Digital network coding improves $R^{avg}$ over plain routing and approaches the rate performance of analog network coding, as $K$ increases. Instead, we can let nodes retransmit packets until successful reception. For all schemes, the probability of

$^5$The analysis follows for any modulation $M$, if (40) holds for $n(x) = n$. 

decoding failure would be reduced to 0 at the expense of increasing the delay, e.g., the delay is increased from $T(K)$ to $\frac{T(K)}{1-\epsilon}$ for the common end-to-end erasure probability $\epsilon$.

V. CONCLUSIONS

In this paper, we compared plain routing, analog and digital network coding for multiple terminals exchanging packets through a single relay. We showed that analog network coding (based on scheduled or random access) improves the delay and throughput over digital network coding for error-free channels. The performance gains over plain routing diminish, as the number of terminals increases, unless terminals overhear each other’s transmissions. For channels with additive noise or packet erasures, we formulated network coding as a multiuser communication problem and illustrated strong trade-offs in terms of the packet delay, achievable rate and reliability.

The broadcast model needs to be extended to multicast communication with arbitrary source-destination pairs. Instead of saturated terminal queues (with always availability of packets to transmit), we may rather allow queues to empty and consider stable operation for random packet arrivals. Future work should also study the scaling effects of packet overhead and control information exchange on the throughput rates.

REFERENCES