1. (a) (10 points) Determine where $e^{iz}$ is analytic (use the Cauchy-Riemann conditions).

(b) (10 points) Find $(-i)^{1/5}$ in polar coordinates for the cut $z$-plane with a branch cut along the negative real axis and $Arg(z) = 0$ along the positive real axis. Just above the cut evaluate $(-1)^{1/5}$.

(c) (10 points) Calculate the Laurent series of the function

$$f(z) = \frac{1}{z^2 + 1}$$

around $z = i$. What is the radius of convergence of this series?

2. (25 points) Evaluate the integral

$$\int_C \frac{dz}{z^2(z^2 + a^2)}$$

where the contour $C$ is a counter-clockwise circle of radius one around the origin and “$a$” is less than one.

3. (20 points) Consider the function $f(x)$ that is periodic over the interval $2L$ and is defined by $f(x) = -1$ for $-L < x < 0$ and $f(x) = 1$ for $0 < x < L$. Define a set of basis functions for functions periodic over the $2L$ interval. Write $f(x)$ in terms of these basis functions.

4. (25 points) Consider the function $f(x) = \frac{1}{x^2 + a^2}$ defined over the interval $-\infty < x < \infty$. Evaluate the Fourier transform $F(k)$ of this function. Write the inverse transform $f(x)$ as an integral over $k$.

Hint: when calculating the transform $F(k)$, consider values of $k > 0$ and $k < 0$ separately.
1a) Determine where $e^{i2}$ is analytic.

$$f(z) = e^z = e^x e^{-y}$$

$$= e^y (\cos x + i \sin x)$$

$$= u + iv$$

$$u = e^y \cos x, \quad v = e^y \sin x$$

$$\frac{\partial u}{\partial x} = -e^y \sin x, \quad \frac{\partial v}{\partial y} = -e^y \sin x$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -e^y \cos x, \quad \frac{\partial v}{\partial x} = -e^y \cos x$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ the C-R conditions are satisfied everywhere $\Rightarrow e^{i2}$ is analytic everywhere.

1b) Find $(-i)^{1/15}$ and $(-i)^{115}$ above the cut. $-1 = e^{i\pi}$.
(1c) Calculate the Laurent series of
\[ f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z-i)(z+i)} \]
around \( z = \frac{\pi}{2} i \). Let \( t = z - i \)
\[ f = \frac{1}{t} \frac{1}{t+2i} = \frac{1}{2it} \frac{1}{1+\frac{t}{2i}} \]
\[ = \frac{1}{2i} \frac{1}{t} \frac{1}{1-i \frac{t}{2}} \]
\[ f = \frac{1}{2i} \frac{1}{t} \sum_{n=0}^{\infty} \left( \frac{it}{2} \right)^n \text{ with } t = z - i \]

Radius of convergence is the distance to the nearest singularity \( \Rightarrow z = -i \)
\( \Rightarrow \) radius of convergence is 16 \( \Rightarrow 2 \).

(2) Evaluate the integral
\[ I = \oint_C \frac{dz}{z^2(z^2 + a^2)} \]
with \( C \) a circle of radius 1 around the origin and \( a < 1 \).
Shrink \( C \) around singularities at \( z = 0 \) and \( z = \pm i a \)

\[
I_0 = \oint_{C_0} \frac{dz}{z^2 (z^2 + a^2)}
\Rightarrow \text{second order pole at } z = 0
\Rightarrow \text{use Cauchy's derivative formula}
\]

\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{C_0} \frac{f(z)}{(z - z_0)^{n+1}}
\]

with \( n = 1 \) and \( f(z) = \frac{1}{z^2 + a^2} \) and \( z_0 = 0 \)

\[
I_0 = \oint_{C_0} \left( \frac{1}{z^2 + a^2} \right)' \bigg|_{z=0} = 2\pi i \left. \frac{2z}{(z^2 + a^2)^2} \right|_{z=0}
= 0
\]

\[
I_+ = \oint_{C_+} \frac{dz}{z^2 (z - ia)(z + ia)} = 2\pi i \left. \frac{1}{(ia)^2 - z a} \right|_{z=ia}
= -\frac{\pi}{a^3}
\]

\[
I_- = \oint_{C_-} \frac{dz}{z^2 (z - ia)(z + ia)} = 2\pi i \left. \frac{1}{(zia)^2 - (-2ia)} \right|_{z=-2ia}
= \frac{\pi}{a^3}
\]

\[
\Gamma = I_0 + I_+ + I_- = 0
\]
Basis functions are

\[
\sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)
\]

Since \(f(x)\) is an odd function around \(x = 0\), only \(\sin\left(\frac{n\pi x}{L}\right)\) contribute.

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)
\]

Multiply by \(\sin\left(\frac{n'\pi x}{L}\right)\) and integrate \((-L, L)\)

\[
\int_{-L}^{L} f(x) \sin\left(\frac{n'\pi x}{L}\right) dx = \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n'\pi x}{L}\right) dx
\]

\[
= \sum_{n=1}^{\infty} b_n \sin' \frac{1}{2} (2L)
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right) dx
\]

since both \(f\) and \(\sin\) are odd and product is even.

\[
b_n = \frac{2}{L} \left[ -\cos\left(\frac{n\pi x}{L}\right) \right]_{0}^{L} = \frac{2}{n\pi} \left( 1 - \cos(\pi) \right)
\]

\[
= \begin{cases} 
\frac{4}{n\pi} & \text{n odd} \\
0 & \text{n even}
\end{cases}
\]
\[ f(x) = \sum_{n=1,3,5,\ldots} \frac{4}{\pi n} \sin\left(\frac{n\pi x}{2}\right) \]

4) \[ f(x) = \frac{1}{x^2 + a^2} \]

Calculate \[ F(k) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dx' \frac{e^{-ikx'}}{x'^2 + a^2} \]

\[ e^{-ikx'} \] is bounded in upper half plane so close contour in LHP. By Jordan's Lemma the contribution from the circle is zero.

\[ F(k) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dx' \frac{e^{-ikx'}}{c' (x'-ia)(x'+ia)} \]

\[ = \frac{i}{2\pi} \left( \frac{e^{-i(k+ia)} - ika}{-2ia} \right) = \frac{e^{-ka}}{2a} \]

\[ e^{-iKx'} \] is bounded in UHP so close contour there.
\[ F(k) = \frac{1}{2\pi} \oint_{c} \frac{e^{-i k x'}}{c'(x' - i a)(x' + i a)} \, dx' \]

\[ = \frac{2\pi i}{2\pi} \frac{e^{-i k a}}{2i a} = \frac{e^{-i k a}}{2a} \]

\[ F(k) = \frac{e^{-i k a}}{2a} \]

\[ f(x) = \int_{-\infty}^{\infty} dk \frac{e^{-i k a}}{2a} e^{i k x} \]