1. (a) Find the solutions of the following equation,

\[ \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} - \alpha y = 0. \]

Show that the solutions are linearly independent. What are the solutions when \( \alpha = -4 \)?

(b) The Spherical Bessel Equation is given by

\[ x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + [x^2 - n(n + 1)]y = 0. \]

What is the form of the differential equation very close to the singularity at \( x = 0 \)? What is the lowest order behavior of the solutions near \( x = 0 \)? Show that these two solutions are linearly independent.

(c) Take the Laplace transform of \( \cos(\omega t) \). If \( \omega = \omega_0 + i\gamma \) is complex such that \( \omega_0 \) and \( \gamma \) are real, what is the requirement on \( p \) for the Laplace transform to be valid?

2. A differential equation is given by

\[ (1 + x^2) \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0. \]

Calculate the series solutions of this equation around \( x = 0 \). What is the radius of convergence of this series?

3. Consider the differential equation

\[ \frac{d^2y}{dt^2} + y = e^{-t}, \]

where the source on the right side of the equation is turned on at \( t = 0 \) and with the initial conditions \( y(0) = 0 \) and \( dy(0)/dt = 0 \).

(a) Calculate the homogeneous and particular solutions of the equation. Construct a linear combination of these to satisfy the initial conditions at \( t = 0 \).

(b) Take the Laplace transform of the equation to obtain an equation for \( Y(p) \). Write down the inverse transform for \( y(t) \) and evaluate the integral in the complex \( p \) plane to obtain \( y(t) \). It should agree with your earlier solution based on the particular and homogeneous solutions.
Physics 373 Midterm #2 Solutions

1) a) Find the solutions of

\[ y'' + 4y' - ay = 0 \]

\[ \Rightarrow \text{constant coefficients} \Rightarrow \text{exponential solutions} \]

\[ y = e^{px} \]

\[ (p^2 + 4p - a)e^{px} = 0 \]

\[ \Rightarrow p^2 + 4p - a = 0 \]

\[ p = -4 \pm \sqrt{16 + 4a} = -2 \pm \sqrt{4 + a} \]

\[ y = e^{px} \]

Linearly independent? \Rightarrow show Wronskian is non-zero

\[ W(0) = \begin{vmatrix} e^{p_+x} & e^{p_-x} \\ p_+e^{p_+x} & p_-e^{p_-x} \end{vmatrix}_{x=0} = \begin{vmatrix} 1 & 1 \\ p_+ & p_- \end{vmatrix} \]

\[ = p_- - p_+ = -2\sqrt{4 + a} \neq 0 \]

\Rightarrow linearly independent

\Rightarrow except \ a = -4 \Rightarrow \text{repeated root} \]

\Rightarrow \text{solutions are} \ e^{-2x}, \ xe^{-2x}
6) Spherical Bessel Eqn

\[ x^2 y'' + 2xy' + \left[ x^2 - n(n+1) \right] y = 0 \]

Near \( x = 0 \),

\[ x^2 y'' + 2xy' - n(n+1)y = 0 \]

\[ \Rightarrow \text{Euler eqn} \Rightarrow \text{power law solutions} \]

\[ y \sim x^r \]

\[ \left[ r(r-1) + 2r - n(n+1) \right] x^r = 0 \]

\[ \Rightarrow \quad n(n+1) = n(n+1) \]

\[ r = n \quad -(n+1) \]

\[ y \sim x^n \quad x^{-(n+1)} \]

\[ \vec{W} = \begin{vmatrix} x^n & x^{-(n+1)} \\ n x^{n-1} & -(n+1) x \end{vmatrix} = x^{-2} \left( -(n+1) - n \right) = -x^{-2} (2n+1) \neq 0 \]

for \( n \neq -1/2 \)
c) Laplace transform of \( \cos(\omega t) \)

with \( \omega = \omega_0 + i \delta \) with \( \omega_0, \delta \) real

\[
L(\cos(\omega t)) = \int_0^\infty e^{-pt} \cos(\omega t) \, dt
\]

\[
= \frac{1}{2} \left( e^{-\omega_0 t} + e^{-(\omega_0 + 2\delta) t} \right)
\]

\[
= \frac{1}{2} \left( \frac{e^{-\omega_0 t}}{\omega_0 - p} + \frac{e^{-\omega_0 t}}{-\omega_0 + p} \right)
\]

\[
= -\frac{1}{2} \left( \frac{1}{\omega_0 - p} - \frac{1}{\omega_0 + p} \right) \text{ for } p > \delta
\]

\[
= -\frac{1}{2} \frac{i\omega + p - (i\omega - p)}{-\omega^2 - p^2} = \frac{p}{\omega^2 + p^2}
\]

Requird \( R(p) > \delta \) so the integral integrant

\( \Rightarrow 0 \text{ as } t \rightarrow \infty \).

2) Diff Eqn

\[
(1 + x^2) y'' + 4x y' + 2y = 0
\]

\( \Rightarrow \) analytic around \( x = 0 \)?
\[ y'' + \frac{4x}{1+x^2} y' + \frac{2}{1+x^2} y = 0 \]

where \( a_0(x), a_1(x) \) are analytic around \( x = 0 \)

\[ \Rightarrow \text{positive series in } x \]

\[ y = \sum_{n=0}^{\infty} c_j x^j \]

\[ \sum_{j=0}^{N_0} \left[ j(j-1)c_j x^{j-2} + j(j-1)c_j x^j \right. \]

\[ + 4j c_j x^j + 2c_j x^j \left. \right] = 0 \]

\[ \sum_{j=0}^{N_0} \left[ j(j-1)c_j x^{j-2} \right. \]

\[ + \underbrace{\left[ j(j-1) + 4j + 2 \right] c_j x^j}_{j^2 + 3j + 2} \]

\[ \underbrace{(j+1)(j+2)}_{(j+1)(j+2)} \]

\[ \sum_{j=0}^{N_0} \left[ j(j-1)c_j x^{j-2} + (j+1)(j+2)c_j x^j \right] = 0 \]

\[ 2 \cdot 1 \cdot C_2 x^0 + 1 \cdot 2 \cdot C_0 x^0 \]

\[ + 3 \cdot 2 \cdot C_3 x^1 + 2 \cdot 3 \cdot C_1 x^1 \]

\[ + \ldots \]

\[ \Rightarrow \text{same powers of } x \text{ for each series} \]
shift left series up by 2 \Rightarrow j \Rightarrow j+2

\rho \sum_{j=0}^{M_{\rho}} \left[ (j+2)j+1 \right] c_{j+2} + (j+1)(j+2) c_j j^j x^j = 0

\Rightarrow 0

C_{j+2} = - \frac{(j+1)(j+2)}{(j+1)(j+2)} c_j = -c_j

C_0 \text{ se ries} ; \quad C_2 = -c_0
\quad C_4 = -c_2 = c_0
\quad C_6 = -c_4 = -c_0

y_0 = c_0 \sum_{j=0}^{M_{\rho}} (-1)^{j/2} x^j
\quad \text{jeven}

C_1 \text{ se ries} ; \quad C_3 = -c_1
\quad C_5 = -c_3 = c_1

y_1 = c_1 \sum_{j=1}^{M_{\rho}} (-1)^{j-1} x^j
\quad \text{jodd}

\text{z-plane}

\text{Radius of convergence is distance to singularity at } \pm i

\Rightarrow |x| < 1
Diff eqn.
\[ y'' + y = e^{-t} \]
RHS turned on at \( t=0 \) with
\[ y(0) = 0, \quad y'(0) = 0 \]

a) Particular solution
\[ y_p = Ae^{-t} \]  \( \Rightarrow \) insert into eqn.
\[ Ae^{-t} (1 + 1) = e^{-t} \]
\[ A = \frac{1}{2} \]  \( \Rightarrow \) \[ y_p = \frac{1}{2} e^{-t} \]

Homogeneous solutions
\[ y'' + y = 0 \]  \( \Rightarrow \) \[ e^{i\omega t} \]
\[ (-\omega^2 + 1) e^{i\omega t} = 0 \] \( \Rightarrow \) \( \omega = \pm 1 \)
\[ y_h = e^{-t} e^{i\omega t} \] on \( \sin(\omega t) \) cos(\omega t)

\[ y = \frac{1}{2} e^{-t} + a \sin(t) + b \cos t \]
\[ y(0) = 0 = \frac{1}{2} + b \]  \( \Rightarrow \) \( b = -\frac{1}{2} \)
\[ y'(0) = -\frac{1}{2} + a \]  \( \Rightarrow \) \( a = \frac{1}{2} \)
6) Take Laplace transform of given with 
\( y(0) = 0, \dot{y}(0) = 0 \)

\[ L(\ddot{y}) + L(y) = L(e^{-t}) \]

\[ \frac{d^2 Y(p)}{dp^2} - Y(0) - pY(0) + Y(p) = L(e^{-t}) \]

\[ (p^2 + 1) Y(p) = L(e^{-t}) \]

\[ L(e^{-t}) = \int_0^{\infty} e^{-t} e^{-st} \, dt = \frac{1}{s + 1} \]

\[ Y(p) = \frac{1}{(p+1)(p^2+1)} \]

\[ y(t) = \frac{1}{2\pi i} \oint_C e^{pt} \frac{dp}{(p+1)(p+i)(p-i)} \]

\[ \Rightarrow \text{shrink around 3 poles} \]

\[ \Rightarrow \text{close on the left for } t > 0 \]

\[ \Rightarrow \text{neglect contour at large } R \]

\[ \Rightarrow \text{Jordan's Lemma} \]
\[ y(t) = \frac{2\pi e^{-\frac{pt}{2}}}{2\pi i} \left[ \frac{e^{\frac{pt}{2}}}{(p+1)(p+i)} \right]_{p=-1}^{p=c} + \frac{e^{\frac{pt}{2}}}{(p+1)(p-i)} \right]_{p=-1}^{p=c} \]

\[ = \frac{e^{\frac{it}{2}}}{2i (1+i)} + \frac{e^{-\frac{it}{2}}}{-2i (1-i)} + \frac{e^{-t}}{2} \]

\[ = \frac{1}{2} e^{-t} + \frac{1}{2i} \left[ e^{i t (1-i)} - e^{i t (1+i)} \right] \]

\[ y(t) = \frac{1}{2} e^{-t} + \frac{1}{2} \left[ \sin t - \cos t \right] \Rightarrow \text{as before} \]