1. (a) (20 points) Carry out the integral
\[ \int_{-\infty}^{\infty} dz \frac{\cos(z)}{z^2 + a^2} \]

(b) (15 points) Evaluate \((-i)^{1/3}\) in the cut \(z\) plane with a branch cut along the negative real axis and \(\text{Arg}(z) = 0\) along the positive real axis. Evaluate \((-1)^{1/3}\) just below the cut.

(c) (20 points) Calculate the Laurent series of the function
\[ f(z) = \frac{1}{z^2 + 1} \]
around \(z = i\).

(d) (20 points) The Bessel Equation is given by
\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + [x^2 - p^2] y = 0. \]
What is the form of the differential equation very close to the singularity at \(x = 0\)? What is the lowest order behavior of the solutions near \(x = 0\)? Show that these two solutions are linearly independent.

2. Consider the differential equation
\[ \frac{d^2 y}{dt^2} + y = \sin(2t), \]
where the source on the right side of the equation is turned on at \(t = 0\) and with the initial conditions \(y(0) = 0\) and \(dy(0)/dt = 0\).

(a) (20 points) Take the Laplace transform of the equation to obtain an equation for \(L(y) = Y(p)\).

(b) (20 points) Write down the inverse transform for \(y(t)\) and evaluate the integral in the complex \(p\) plane to obtain \(y(t)\).

3. The Legendre Polynomials \(P_l(x)\) form a complete set over the interval \(x \in (-1, 1)\).
(a) (15 points) Use fact that $P_0(1) = 1$ and $P_1(1) = 1$ and the recursion formula
\[ lP_l(x) = (2l - 1)xP_{l-1}(x) - (l - 1)P_{l-2}(x) \]
to evaluate $P_l(1)$ for all $l$. What is $P_l(-1)$ for $l$ even? For $l$ odd?

(b) (20 points) The function $f(x)$ is $-1$ for $x \in (-1,0)$ and $1$ for $x \in (0,1)$. Express $f(x)$ as an infinite sum of Legendre polynomials.
Hint: Use symmetry arguments to simplify the problem and the recursion formula
\[ (2l + 1)P_l(x) = P'_{l+1} - P'_{l-1} \]
to carry out some of the integration. You can leave your answer in terms of $P_l(0)$, which is a known function.

4. Consider the heat conduction in an infinitely long cylindrical solid object of radius $R$ in which the boundary temperature is maintained at zero. The equation for the temperature $T(r, t)$ is given by
\[ \frac{\partial T}{\partial t} - D \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} T = 0, \]
where at $t = 0$ the temperature is initially uniform $T(r, t = 0) = T_0$. Express the space-time dependence of $T$ in terms of a set of basis functions as
\[ T(r, t) = \sum_n C_n R_n(r) T_n(t). \]

(a) (15 points) Calculate the equation satisfied by $R_n(r)$, define the boundary condition satisfied by $R_n$, the eigenvalues and the orthogonality condition.

(b) (10 points) Write down the equation satisfied by $T_n(t)$ and its solution.

(c) (15 points) Using the initial condition for $T$ at $t = 0$, calculate the coefficients $C_n$ and write the full solution for $T(r, t)$.
Hint: the recursion formula
\[ \frac{d}{dx} [x^n J_p(x)] = x^n J_{p-1}(x) \]
will be helpful.

(d) (10 points) At late time what is the approximate solution for $T(r, t)$? What is the characteristic decay time of the temperature in the cylinder at this late time?