Math 246, Second In-Class Exam Solutions (Spring 2003)

Professor Levermore

(1) (12 points) Let $L$ be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (shown with multiplicities) are $-3 + i4$, $-3 + i4$, $-3 - i4$, $-3 - i4$, $i5$, $-i5$, $-7$, $-7$, $0$, $0$.

(a) What is the order of $L$?

Solution: There are ten roots listed, so the degree of the characteristic polynomial is ten, and consequently the order of $L$ must be ten.

(b) Give a general real solution of the homogeneous equation $Ly = 0$?

Solution: The general solution is

$$y = c_1 e^{-3t} \cos(4t) + c_2 e^{-3t} \sin(4t) + c_3 t e^{-3t} \cos(4t) + c_4 t e^{-3t} \sin(4t) + c_5 \cos(5t) + c_6 \sin(5t) + c_7 e^{-7t} + c_8 t e^{-7t} + c_9 + c_{10} t.$$ 

The reasoning is as follows.

- The double conjugate pair $-3 \pm i4$ yields $e^{-3t} \cos(4t)$, $e^{-3t} \sin(4t)$, $t e^{-3t} \cos(4t)$, and $t e^{-3t} \sin(4t)$.
- The conjugate pair $\pm i5$ yields $\cos(5t)$ and $\sin(5t)$.
- The double real root $-7$ yields $e^{-7t}$ and $t e^{-7t}$.
- The double real root $0$ yields $1$ and $t$.

(2) (9 points) Solve the initial-value problem

$$y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$ 

Solution: This is a constant coefficient, homogeneous linear initial-value problem. It may be either (1) solved by first finding the general solution or (2) solved directly using the Laplace transform.

The characteristic polynomial is

$$P(z) = z^2 - 6z + 9 = (z - 3)^2.$$ 

It has the double real root $3$, which yields the general solution

$$y(t) = c_1 e^{3t} + c_2 t e^{3t}.$$ 

Because

$$y'(t) = 3c_1 e^{3t} + c_2 (e^{3t} + 3t e^{3t}),$$

when the initial conditions are imposed, one finds that

$$y(0) = c_1 = 0, \quad y'(0) = 3c_1 + c_2 = 1.$$ 

These are solved to find that $c_1 = 0$ and $c_2 = 1$. The solution of the initial-value problem is therefore

$$y(t) = t e^{3t}.$$
We now show how to solve the problem using the Laplace transform. The Laplace transform of the initial-value problem is
\[ \mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = 0, \]
where
\[ \mathcal{L}\{y\}(s) = Y(s), \]
\[ \mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s), \]
\[ \mathcal{L}\{y''\}(s) = s^2Y(s) - s y(0) - y'(0) = s^2Y(s) - 1. \]
The Laplace transform of the initial-value problem then becomes
\[ s^2Y(s) - 1 - 6s Y(s) + 9Y(s) = 0, \]
which can be put in the form
\[ (s^2 - 6s + 9)Y(s) = 1. \]
Upon solving this for \( Y(s) \) one finds that
\[ Y(s) = \frac{1}{s^2 - 6s + 9} = \frac{1}{(s - 3)^2}. \]
Referring to the table on the last page, Item 1 with \( n = 1 \) gives \( \mathcal{L}\{t\} = 1/s^2 \).
Item 4 with \( a = 3 \) and \( f(t) = t \) then gives
\[ \mathcal{L}\{e^{3t}\} = \frac{1}{(s - 3)^2}. \]
One thereby concludes that
\[ y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s - 3)^2} \right\}(t) = t e^{3t}. \]

(3) (27 points) Find a general solution for each of the following equations.
(a) \( y'' + 16y = 5e^{3t}. \)

**Solution:** This is a constant coefficient, nonhomogeneous linear problem. The characteristic polynomial of its homogeneous part is
\[ P(z) = z^2 + 16. \]
It has the simple complex pair of roots \(-i4\) and \(i4\), which yields the general homogeneous solution
\[ y_h(t) = c_1 \cos(4t) + c_2 \sin(4t). \]
Because the forcing is of the form \( e^{zt} \) for \( z = 3 \), and because \( z = 3 \) is not a root of the characteristic polynomial, a particular solution can be found quickly by either the method of undetermined coefficients (as in the book) or the method of determined coefficients (as in class).
The method of **undetermined coefficients** seeks a particular solution of the form \( y_p(t) = Ae^{3t} \). Because
\[ y_p(t) = Ae^{3t}, \quad y'_p(t) = 3Ae^{3t}, \quad y''_p(t) = 9Ae^{3t}, \]
one sees that
\[ Ly_p = y''_p + 16y_p = (9 + 16)Ae^{3t} = 25Ae^{3t} = 5e^{3t}, \]
which implies \( A = 1/5 \). Hence, \( y_p(t) = \frac{1}{5}e^{3t}. \)
The method of *determined coefficients* evaluates the identity
\[ L(e^{zt}) = (z^2 + 16)e^{zt}, \]
at \( z = 3 \) to obtain \( L(e^{3t}) = 25e^{3t} \). Dividing this by 5 gives \( L\left(\frac{1}{5}e^{3t}\right) = \frac{1}{5}e^{3t} \).

By either method you find the same \( y_p \), and the general solution of the problem is therefore
\[ y(t) = c_1 \cos(4t) + c_2 \sin(4t) + \frac{1}{5}e^{3t}. \]

(b) \( y'' + 4y' + 8y = 6 \sin(2t) \).

**Solution:** This is a constant coefficient, nonhomogeneous linear problem. The characteristic polynomial of its homogeneous part is
\[ P(z) = z^2 + 4z + 8 = (z + 2)^2 + 4. \]
It has the conjugate pair of roots \(-2 \pm i2\), which yields the general homogeneous solution
\[ y_h(t) = c_1 e^{-2t} \cos(2t) + c_2 e^{-2t} \sin(2t). \]

Because the forcing is of the form \( e^{zt} \) for \( z = i2 \), and because \( z = i2 \) is not a root of the characteristic polynomial, a particular solution can be found quickly by either the method of undetermined coefficients (as in the book) or the method of determined coefficients (as in class).

The method of *undetermined coefficients* seeks a particular solution of the form \( y_p(t) = A \cos(2t) + B \sin(2t) \). Because
\[ y_p(t) = A \cos(2t) + B \sin(2t), \]
\[ y'_p(t) = -2A \sin(2t) + 2B \cos(2t), \]
\[ y''_p(t) = -4A \cos(2t) - 4B \sin(2t), \]
on one sees that
\[ Ly_p = y''_p + 4y'_p + 8y_p \]
\[ = (4A + 8B) \cos(2t) + (4B - 8A) \sin(2t) = 6 \sin(2t). \]

This leads to the algebraic linear system
\[ 4A + 8B = 0, \quad 4B - 8A = 6. \]
This can be solved to find that \( A = -3/5 \) and \( B = 3/10 \). Hence,
\[ y_p(t) = -\frac{3}{5} \cos(2t) + \frac{3}{10} \sin(2t). \]

The method of *determined coefficients* evaluates the KEY identity \( L e^{zt} = (z^2 + 4z + 8)e^{zt} \) at \( z = i2 \) to obtain \( L e^{2it} = (4 + i8)e^{2it} \). Multiplying this by \( 6/(4 + i8) \) shows that
\[ L \left(\frac{6}{4 + i8} e^{2it}\right) = 6e^{2it}. \]
Because $6e^{i2t} = 6 \cos(2t) + i6 \sin(2t)$, the imaginary part of the left-hand side above will be $Ly_P$. Because

$$
\frac{6}{4 + i8} e^{i2t} = \frac{6}{4 + i8} \frac{4 - i8}{4 - i8} e^{i2t} = \frac{6(4 - i8)}{4^2 + 8^2} e^{i2t} = \frac{6(4 - i8)}{80} (\cos(2t) + i\sin(2t)) = \left(\frac{24}{80} \cos(2t) + \frac{48}{80} \sin(2t)\right) + i\left(-\frac{48}{80} \cos(2t) + \frac{24}{80} \sin(2t)\right),
$$

this imaginary part shows that

$$
y_P(t) = \frac{-48}{80} \cos(2t) + \frac{24}{80} \sin(2t) = -\frac{3}{5} \cos(2t) + \frac{3}{10} \sin(2t).
$$

By either method you find the same $y_P$, and the general solution of the problem is therefore

$$
y = c_1e^{-2t} \cos(2t) + c_2e^{-2t} \sin(2t) - \frac{3}{5} \cos(2t) + \frac{3}{10} \sin(2t).
$$

(c) $y'' + 2y' - 3y = e^t$.

**Solution:** This is a constant coefficient, nonhomogeneous linear problem. The characteristic polynomial of its homogeneous part is

$$
P(z) = z^2 + 2z - 3 = (z - 1)(z + 3).
$$

It the simple real roots $-3$ and $1$, which yields the general homogeneous solution

$$
y_H(t) = c_1e^{-3t} + c_2e^t.
$$

Because the forcing is of the form $e^{zt}$ for $z = 1$, and because $z = 1$ is a root of the characteristic polynomial, a particular solution can be found quickly by either the method of undetermined coefficients (as in the book) or the method of determined coefficients (as in class).

The method of **undetermined coefficients** seeks a particular solution of the form $y_p(t) = Ate^t$. Because

$$
y_p'(t) = A(e^t + t e^t), \quad y_p''(t) = A(2e^t + t e^t),
$$

one sees that

$$
Ly_p = y_p'' + 2y_p' - 3y_p = A(2e^t + t e^t) + 2A(e^t + t e^t) - 3t e^t = A4e^t = e^t,
$$

which implies $A = 1/4$. Hence, $y_p(t) = \frac{1}{4}te^t$.

The method of **determined coefficients** evaluates the identity

$$
L(te^{zt}) = (z^2 + 2z - 3)te^{zt} + (2z + 2)e^{zt},
$$

at $z = 1$ to obtain $L(te^t) = 4e^t$. Dividing this by 4 gives $L(\frac{1}{4}te^t) = e^t$, which shows that $y_p(t) = \frac{1}{4}te^t$. 
By either method you find the same \( y_p \), and the general solution of the problem is therefore
\[
y(t) = c_1 e^{-t} + c_2 e^t + \frac{1}{4} t e^t.
\]

(4) (9 points) The functions \( 1 + x \) and \( e^x \) are solutions of the equation
\[
xy'' - (1 + x)y' + y = 0, \quad x > 0.
\]
(You do not have to check that this is true.)

(a) Compute their Wronskian.

**Solution:** The Wronskian \( W(x) \) of \( 1 + x \) and \( e^x \) is given by
\[
W(x) = \det \begin{pmatrix} 1 + x & e^x \\ 1 & e^x \end{pmatrix} = (1 + x)e^x - e^x = xe^x.
\]
Note \( W(x) > 0 \) when \( x > 0 \), so \( 1 + x \) and \( e^x \) are linearly independent.

(b) Find a general solution of the equation
\[
xy'' - (1 + x)y' + y = x^2 e^x, \quad x > 0.
\]

**Solution:** The general solution of this nonhomogeneous equation will have the form \( y = y_H + y_P \), where \( y_H \) is the general solution of the corresponding homogeneous equation and \( y_P \) is any particular solution of the nonhomogeneous equation. Because you are given that \( 1 + x \) and \( e^x \) are solutions of the corresponding homogeneous equation, and you know by part (a) that they are linearly independent, you know that
\[
y_H = c_1 (1 + x) + c_2 e^x.
\]
The methods of undetermined or determined coefficients cannot be used to find a particular solution, so we will use the method of variation of parameters. We first put the equation into its normal form
\[
y'' = \frac{1 + x}{x} y' + \frac{1}{x} y = xe^x,
\]
and then seek \( y_P \) of the form
\[
y_P = (1 + x)u_1(x) + e^x u_2(x).
\]
One chooses \( u'_1 \) and \( u'_2 \) so that they satisfy
\[
(1 + x)u'_1 + e^x u'_2 = 0, \quad u'_1 + e^x u'_2 = xe^x.
\]
This linear system is solved to find that
\[
u'_1 = -e^x, \quad u'_2 = 1 + x.
\]
Upon integrating these, you find that
\[
u_1(x) = c_1 - e^x, \quad u_2(x) = c_2 + x + \frac{1}{2} x^2.
\]
Your answer can be expressed as
\[
y = c_1 (1 + x) + c_2 e^x - (1 + x)e^x + e^x(x + \frac{1}{2} x^2).
\]
This can be simplified to
\[
y = c_1 (1 + x) + c_3 e^x + \frac{1}{2} x^2 e^x,
\]
where \( c_3 = c_2 - 1 \).
(5) (6 points) The vertical displacement of a mass on a spring is given by

\[ z(t) = 4 \cos(7t) + 3 \sin(7t) . \]

Express this in the form \( z(t) = A \cos(\omega t - \delta) \), identifying the amplitude and phase of the oscillation.

**Solution:** The displacement takes the form

\[ z(t) = 5 \cos \left( 7t - \tan^{-1} \left( \frac{3}{4} \right) \right) , \]

where the amplitude is 5, the frequency is 7, and the phase is \( \tan^{-1} \left( \frac{3}{4} \right) \).

There are several approaches to this problem. Here are two.

One approach that requires no memorization other than the usual addition formula for cosine is as follows. Because

\[ A \cos(\omega t - \delta) = A \cos(\delta) \cos(\omega t) + A \sin(\delta) \sin(\omega t) , \]

this form will be equal to \( y(t) \) provided \( \omega = 7 \) and

\[ A \cos(\delta) = 4 , \quad A \sin(\delta) = 3 . \]

Upon solving these equations one finds that the amplitude \( A \) is given by

\[ A = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5 , \]

while the phase \( \delta \) is given either by

\[ \delta = \sin^{-1} \left( \frac{3}{5} \right) = \sin^{-1} \left( \frac{3}{5} \right) , \]

or by

\[ \delta = \cos^{-1} \left( \frac{4}{5} \right) = \cos^{-1} \left( \frac{4}{5} \right) , \]

or by

\[ \delta = \tan^{-1} \left( \frac{3}{4} \right) . \]

Another approach requires you to memorize special formulas for both the amplitude and phase of functions of the form

\[ c_1 \cos(\omega t) + c_2 \sin(\omega t) . \]

The formula for the amplitude is easier one because \( c_1 \) and \( c_2 \) appear in it symmetrically. It gives

\[ A = \sqrt{c_1^2 + c_2^2} = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5 . \]

The formula for the phase is trickier because \( c_1 \) and \( c_2 \) do not appear in it symmetrically. It gives

\[ \delta = \tan^{-1} \left( \frac{c_2}{c_1} \right) = \tan^{-1} \left( \frac{3}{4} \right) . \]

The most common mistake made by those who chose this approach was to exchange the roles of \( c_1 \) and \( c_2 \) in this formula. One way to keep these roles straight is to remember the formula verbally as

\[ \text{phase} = \tan^{-1} \left( \frac{\text{coefficient of sine}}{\text{coefficient of cosine}} \right) . \]
(6) (10 points) When a 2 kilogram (kg) mass is hung vertically from a spring, at rest it stretches the spring .2 meters (m). (Gravitational acceleration is \( g = 9.8 \text{ m/sec}^2 \).) At \( t = 0 \) the mass is displaced .1 m above its equilibrium position and released with no initial velocity. It moves in a medium that imparts a drag force of 4 Newtons (1 Newton = 1 kg m/sec^2) when the speed of the mass is 5 m/sec. There are no other forces. (As usual, assume the spring force is proportional to displacement and the drag force is proportional to velocity.)

(a) Formulate an initial-value problem that governs the motion of the mass for \( t > 0 \). (DO NOT solve the initial-value problem, just write it down!)

**Solution:** Let \( y \) be the displacement of the mass from the equilibrium position in meters, with upward displacements being positive. The governing initial-value problem then has the form

\[
my'' + \gamma y' + ky = 0, \quad y(0) = .1, \quad y'(0) = 0,
\]

where \( m \) is the mass, \( \gamma \) is the drag coefficient, and \( k \) is the spring constant. The problem says that \( m = 2 \) kilograms. The spring constant is obtained by balancing the weight of the mass (\( mg = 2 \cdot 9.8 \) Newtons) with the force applied by the spring when it is stretched .2 meters. This gives

\[
k = \frac{2 \cdot 9.8}{2} = 10 \cdot 9.8 = 98 \text{ kg/sec}^2.
\]

The drag coefficient is obtained by balancing the force of 4 Newtons with the drag force imparted by the medium when the speed of the mass is 5 m/sec. This gives \( \gamma \gamma 5 = 4 \), or

\[
\gamma = \frac{4}{5} \text{ kg/sec}.
\]

The governing initial-value problem is therefore

\[
2y'' + \frac{4}{5}y' + 98y = 0, \quad y(0) = .1, \quad y'(0) = 0.
\]

If you had chosen downward displacements to be positive then the governing initial-value problem would be identical except for the first initial condition, which would then be \( y(0) = -.1 \).

(b) Give the natural frequency of the spring.

**Solution:** The natural frequency of the spring is given by

\[
\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{98}{2}} = \sqrt{49} = 7 \text{ 1/sec}.
\]

(c) Show that the system is under-damped and give its quasifrequency.

**Solution:** The characteristic polynomial is

\[
P(z) = z^2 + \frac{2}{5}z + 49 = \left(z + \frac{1}{5}\right)^2 + 49 - \frac{1}{25},
\]

which has the complex roots

\[
z = -\frac{1}{5} \pm i\sqrt{49 - \frac{1}{25}}.
\]
The system is therefore under-damped with a quasifrequency $\mu$ given by
$$
\mu = \sqrt{49 - \frac{1}{25}}.
$$

(7) (6 points) Compute the Laplace transform of $f(t) = e^{-4t}$ from its definition.

**Solution:** Let $F(s) = \mathcal{L}\{f\}(s)$. By the definition of the Laplace transform
$$
F(s) = \int_0^\infty e^{-st} e^{-4t} \, dt = \lim_{M \to \infty} \int_0^M e^{-(s+4)t} \, dt.
$$

For $s + 4 \neq 0$ one has
$$
\int_0^M e^{-(s+4)t} \, dt = \left( -\frac{e^{-(s+4)t}}{s+4} \right)_0^M = \left[ -\frac{e^{-(s+4)M}}{s+4} + \frac{1}{s+4} \right],
$$
while for $s + 4 = 0$ one has
$$
\int_0^M e^{-(s+4)t} \, dt = \int_0^M 1 \, dt = M.
$$

one thereby sees that
$$
F(s) = \lim_{M \to \infty} \left\{ \begin{array}{ll}
-\frac{e^{-(s+4)M}}{s+4} + \frac{1}{s+4} & \text{for } s + 4 \neq 0 \\
\frac{1}{s+4} & \text{for } s + 4 = 0
\end{array} \right.
$$

Hence, one finds that
$$
\mathcal{L}\{e^{-4t}\}(s) = \frac{1}{s+4} \quad \text{for } s > -4.
$$

(8) (9 points) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem
$$
y'' + 9y = f(t), \quad y(0) = 4, \quad y'(0) = 1,
$$
where
$$
f(t) = \begin{cases} 
0 & \text{for } 0 \leq t < 2\pi, \\
t - 2\pi & \text{for } t \geq 2\pi.
\end{cases}
$$

You may refer to the table below. (DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$.)

**Solution:** The Laplace transform of the initial-value problem is
$$
\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{f\},
$$
where
$$
\mathcal{L}\{y\} = Y(s), \quad \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 4, \quad \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s4 - 1.
$$

To compute $\mathcal{L}\{f\}$, first rewrite $f$ as
$$
f(t) = u(t-2\pi)(t-2\pi).
$$
Referring to the table on the last page, Item 5 with \( c = 2 \) and \( f(t) = t \) followed Item 1 with \( n = 1 \) then shows that

\[
\mathcal{L}\{f\} = \mathcal{L}\{u(t - 2\pi)(t - 2\pi)\} = e^{-2\pi s}\mathcal{L}\{t\}(s) = e^{-2\pi s} \frac{1}{s^2}.
\]

The Laplace transform of the initial-value problem then becomes

\[
(s^2 Y(s) - 4s - 1) + 9Y(s) = e^{-2\pi s} \frac{1}{s^2},
\]

which becomes

\[
(s^2 + 9)Y(s) - (4s + 1) = e^{-2\pi s} \frac{1}{s^2}.
\]

Hence, \( Y(s) \) is given by

\[
Y(s) = \frac{1}{s^2 + 9} \left( 4s + 1 + e^{-2\pi s} \frac{1}{s^2} \right).
\]

(9) (12 points) Find the inverse Laplace transform of the following functions:

(a) \( F(s) = \frac{4s}{s^2 - 4} \).

**Solution:** The denominator factors as \((s - 2)(s + 2)\) so the partial fraction decomposition is

\[
F(s) = \frac{4s}{s^2 - 4} = \frac{4s}{(s - 2)(s + 2)} = \frac{2}{s - 2} + \frac{2}{s + 2}.
\]

Referring to the table on the last page, Item 1 with \( n = 0 \) gives \( \mathcal{L}\{1\} = 1/s \). Item 4 with \( a = 2, \ a = -2 \) and \( f(t) = 1 \) then gives

\[
\mathcal{L}\{e^{2t}\} = \frac{1}{s - 2}, \quad \mathcal{L}\{e^{-2t}\} = \frac{1}{s + 2}.
\]

One therefore finds that

\[
\mathcal{L}^{-1}\left\{ \frac{4s}{s^2 - 4} \right\} = 2e^{2t} + 2e^{-2t}.
\]

(b) \( F(s) = \frac{6se^{-5s}}{s^2 + 9} \).

**Solution:** Referring to the table on the last page, Item 2 with \( b = 3 \) gives

\[
\mathcal{L}\{\cos(3t)\} = \frac{s}{s^2 + 9}.
\]

Item 5 with \( c = 5 \) and \( f(t) = 6\cos(3t) \) then gives

\[
\mathcal{L}\{u(t - 5)\cos(3(t - 5))\} = e^{-5s} \frac{6s}{s^2 + 9}.
\]

One therefore finds that

\[
\mathcal{L}^{-1}\left\{ \frac{6se^{-5s}}{s^2 + 9} \right\} = u(t - 5)\cos(3(t - 5)).
\]
A Short Table of Laplace Transforms

\[ \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \]  \quad \text{for } s > 0 .

\[ \mathcal{L}\{\cos(bt)\} = \frac{s}{s^2 + b^2} \]  \quad \text{for } s > 0 .

\[ \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2 + b^2} \]  \quad \text{for } s > 0 .

\[ \mathcal{L}\{e^{at}f(t)\} = F(s - a) \]  \quad \text{where } F(s) = \mathcal{L}\{f(t)\} .

\[ \mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s) \]  \quad \text{where } F(s) = \mathcal{L}\{f(t)\} \\
\text{and } u \text{ is the step function} .