(1) [6] Given that 2 is an eigenvalue of the matrix

\[
A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix},
\]

find all the eigenvectors of \( A \) associated with 2.

**Solution.** The eigenvectors of \( A \) associated with 2 are all nonzero vectors \( v \) such that \( Av = 2v \). Equivalently, they are all nonzero vectors \( v \) such that \((A - 2I)v = 0\), which is

\[
\begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

The entries of \( v \) thereby satisfy the homogeneous linear algebraic system

\[
v_2 - 3v_3 = 0,
\]

\[
v_1 + 2v_3 = 0,
\]

\[2v_1 + v_2 + v_3 = 0.
\]

This system may be solved either by elimination or by row reduction. By either method its general solution is found to be

\[
v_1 = -2\alpha, \quad v_2 = 3\alpha, \quad v_3 = \alpha, \quad \text{for any constant } \alpha.
\]

Therefore the eigenvectors of \( A \) associated with 2 have the form

\[
\alpha \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad \text{for any constant } \alpha \neq 0.
\]

(2) [8] A 3 \times 3 matrix \( A \) has the eigenpairs

\[
(4, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}), \quad (2, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}), \quad (-3, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}).
\]

(a) Give an invertible matrix \( V \) and a diagonal matrix \( D \) such that \( e^{tA} = Ve^{tD}V^{-1} \).

(You do not have to compute either \( V^{-1} \) or \( e^{tA} \).)

(b) Give a fundamental matrix for the system \( x' = Ax \).

**Solution (a).** One choice for \( V \) and \( D \) is

\[
V = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}.
\]
Solution (b). Use the given eigenpairs to construct the special solutions

\[
\begin{align*}
x_1(t) &= e^{4t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, & x_2(t) &= e^{2t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, & x_3(t) &= e^{-3t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},
\end{align*}
\]

Then a fundamental matrix for the system is

\[
\Psi(t) = \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) \end{pmatrix} = \begin{pmatrix} e^{4t} & e^{2t} & e^{-3t} \\ 0 & -e^{2t} & 2e^{-3t} \\ -e^{4t} & e^{2t} & e^{-3t} \end{pmatrix}.
\]

Alternative Solution (b). Given the \( \Psi \) and \( D \) from part (a), a fundamental matrix for the system is

\[
\Psi(t) = V e^{tD} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^{4t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix} = \begin{pmatrix} e^{4t} & e^{2t} & e^{-3t} \\ 0 & -e^{2t} & 2e^{-3t} \\ -e^{4t} & e^{2t} & e^{-3t} \end{pmatrix}.
\]

(3) [6] Recast the equation \( v''' = \sin(v)v'' + (v')^2 - e^tv' \) as a first-order system of ordinary differential equations.

Solution. Because the equation is fourth order, the first-order system must have dimension four. The simplest such first-order system is

\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \sin(x_1)x_4 + (x_3)^2 - e^tx_2 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} v \\ v' \\ v'' \\ v''' \end{pmatrix}.
\]

(4) [12] Consider the vector-valued functions \( x_1(t) = \begin{pmatrix} t^5 + 1 \\ t^2 \\ 1 \end{pmatrix}, \ x_2(t) = \begin{pmatrix} t^3 \\ 1 \end{pmatrix} \).

(a) [2] Compute the Wronskian \( W[x_1, x_2](t) \).

(b) [4] Find \( A(t) \) such that \( x_1, x_2 \) is a fundamental set of solutions to the system \( x' = A(t)x \) whenever \( W[x_1, x_2](t) \neq 0 \).

(c) [2] Give a general solution to the system you found in part (b).

(d) [4] Find the natural fundamental matrix associated with the initial time \(-1\) for the system you found in part (b).

Solution (a). The Wronskian is

\[
W[x_1, x_2](t) = \det \begin{pmatrix} t^5 + 1 & t^3 \\ t^2 & 1 \end{pmatrix} = (t^5 + 1) \cdot 1 - t^2 \cdot t^3 = t^5 + 1 - t^5 = 1.
\]

Solution (b). Let \( \Psi(t) = \begin{pmatrix} t^5 + 1 & t^3 \\ t^2 & 1 \end{pmatrix} \). Because \( \Psi'(t) = A(t)\Psi(t) \), we have

\[
A(t) = \Psi'(t)\Psi(t)^{-1} = \begin{pmatrix} 5t^4 & 3t^2 \\ 2t & 0 \end{pmatrix} \begin{pmatrix} t^5 + 1 & t^3 \\ t^2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 5t^4 & 3t^2 \\ 2t & 0 \end{pmatrix} \begin{pmatrix} 1 & -t^3 \\ -t^2 & 1 + t^5 \end{pmatrix} = \begin{pmatrix} 2t^4 & 3t^2 - 2t^7 \\ 2t & -2t^4 \end{pmatrix}.
\]
Solution (c). A general solution is
\[ x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 \left( \frac{t^5 + 1}{t^2} \right) + c_2 \left( \frac{t^3}{1} \right). \]

Solution (d). By using the fundamental matrix \( \Psi(t) \) from part (b) we find that the natural fundamental matrix associated with the initial time \(-1\) is
\[ \Phi(t) = \Psi(t) \Psi(-1)^{-1} = \left( \begin{array}{cc} t^5 + 1 & t^3 \\ t^2 & 1 \end{array} \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 1 \end{array} \right)^{-1} \]
\[ = \left( \begin{array}{cc} t^5 + 1 & t^3 \\ t^2 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ -1 & 0 \end{array} \right) = \left( \begin{array}{cc} t^5 - t^3 + 1 & t^5 + 1 \\ t^2 - 1 & t^2 \end{array} \right). \]

(5) [8] Let \( A = \begin{pmatrix} 3 & 5 \\ -2 & 1 \end{pmatrix} \).

(a) [2] Find the eigenvalues of \( A \).
(b) [6] Find an eigenvector for each eigenvalue.

Solution (a). The characteristic polynomial of \( A \) is
\[ p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 - 4z + 13 = (z - 2)^2 + 3^2. \]
The eigenvalues of \( A \) are the roots of this polynomial, which are \( 2 + 3i \) and \( 2 - 3i \).

Solution (b). After checking that the determinant is zero for either of the matrices
\[ A - (2 + 3i)I = \begin{pmatrix} 1 - 3i & 5 \\ -2 & -1 - 3i \end{pmatrix}, \quad A - (2 - 3i)I = \begin{pmatrix} 1 + 3i & 5 \\ -2 & -1 + 3i \end{pmatrix}, \]
we can read off from the first columns above that eigenpairs of \( A \) are
\[ \left( 2 + 3i, \begin{pmatrix} 1 + 3i \\ -2 \end{pmatrix} \right), \quad \left( 2 - 3i, \begin{pmatrix} 1 - 3i \\ -2 \end{pmatrix} \right). \]

Remark. We can read off from the second columns above that eigenpairs of \( A \) are
\[ \left( 2 + 3i, \begin{pmatrix} 5 \\ -1 + 3i \end{pmatrix} \right), \quad \left( 2 - 3i, \begin{pmatrix} 5 \\ -1 - 3i \end{pmatrix} \right). \]

Alternative Solution (b). After checking that the determinant is zero for the associated matrix, an eigenvector \( v \) for the eigenvalue \( 2 + 3i \) satisfies
\[ 0 = (A - (2 + 3i)I)v = \begin{pmatrix} 1 - 3i & 5 \\ -2 & -1 - 3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \]
The entries of \( v \) thereby satisfy the homogeneous linear algebraic system
\[ (1 - 3i)v_1 + 5v_2 = 0, \quad 2v_1 + (1 + 3i)v_2. \]
A general solution of this system can be expressed either as
\[ v = \alpha \begin{pmatrix} -5 \\ 1 - 3i \end{pmatrix} \quad \text{or as} \quad v = \alpha \begin{pmatrix} 1 + 3i \\ -2 \end{pmatrix}. \]
This is an eigenvector of \( 2 + 3i \) when \( \alpha \neq 0 \). Its conjugate is an eigenvector of \( 2 - 3i \).
(6) [16] Find a general solution for each of the following systems.

(a) \( \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \)

Solution. The characteristic polynomial of \( A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \) is
\[
p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 - 4z + 4 = (z - 2)^2.
\]
The eigenvalues of \( A \) are the roots of this polynomial, which is only 2. Because \( e^{tA} = e^{2t} \left[ I + t (A - 2I) \right] \)
\[
e^{2t} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \right] = e^{2t} \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix},
\]
(check that \( A - 2I \) has trace zero) a general solution is
\[
x(t) = e^{tA}c = c_1e^{2t} \begin{pmatrix} 1-t \\ -t \end{pmatrix} + c_2e^{2t} \begin{pmatrix} t \\ 1+t \end{pmatrix}.
\]

Alternative Solution. The characteristic polynomial of \( A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \) is
\[
p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 - 4z + 4 = (z - 2)^2.
\]
The eigenvalues of \( A \) are the roots of this polynomial, which is only \(-2\). Because
\[
A - 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},
\]
we can read off that an eigenpair is
\[
\left( 2, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) .
\]
We can use this eigenpair to construct the solution
\[
x_1(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]
A second solution can be constructed by the formula
\[
x_2(t) = e^{tA}w = e^{2t} \left[ I + t (A - 2I) \right]w,
\]
where \( w \) is any nonzero vector that is not an eigenvector for the eigenvalue 2. For example, we can set
\[
x_2(t) = e^{2t} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{2t} \begin{pmatrix} t \\ 1+t \end{pmatrix}.
\]
In that case a general solution is
\[
x(t) = c_1x_1(t) + c_2x_2(t) = c_1e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{2t} \begin{pmatrix} t \\ 1+t \end{pmatrix}.
\]
(b) \( \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \)

**Solution.** The characteristic polynomial of \( A = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix} \) is

\[ p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 - 2z - 15 = (z + 3)(z - 5). \]

The eigenvalues of \( A \) are the roots of this polynomial, which are \(-3\) and \(5\). After checking that the columns are proportional for each of the matrices

\[ A + 3I = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}, \quad A - 5I = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}, \]

we can read off that eigenpairs are

\[ (-3, \begin{pmatrix} -1 \\ 2 \end{pmatrix}), \quad (5, \begin{pmatrix} 3 \\ 2 \end{pmatrix}). \]

Therefore a general solution of the system is

\[ x(t) = c_1 e^{-3t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \]

**Alternative Solution.** The characteristic polynomial of \( A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \) is

\[ p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 - 2z - 15 = (z + 3)(z - 5). \]

The eigenvalues of \( A \) are the roots of this polynomial, which are \(-3\) and \(5\). The average of these eigenvalues is \(1\), so they can be expressed as \(1 \pm 4\). Because

\[ e^{tA} = e^t \left[ \cosh(4t)I + \frac{\sinh(4t)}{4}(A - I) \right] \]

\[ = e^t \left[ \cosh(4t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(4t)}{4} \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \right] \]

\[ = e^t \begin{pmatrix} \cosh(4t) + \frac{1}{2} \sinh(4t) & \frac{3}{4} \sinh(4t) \\ \frac{3}{2} \sinh(4t) & \cosh(4t) - \frac{1}{2} \sinh(4t) \end{pmatrix}, \]

(check that \( A - I \) has trace zero) a general solution is

\[ x(t) = e^{tA}c = c_1 e^t \begin{pmatrix} \cosh(4t) + \frac{1}{2} \sinh(4t) \\ \frac{3}{2} \sinh(4t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \frac{3}{4} \sinh(4t) \\ \cosh(4t) - \frac{1}{2} \sinh(4t) \end{pmatrix}. \]

(7) [10] Solve the initial-value problem

\[ \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]

**Solution.** The characteristic polynomial of \( A = \begin{pmatrix} 1 & 1 \\ -4 & -1 \end{pmatrix} \) is

\[ p(z) = z^2 - \text{tr}(A)z + \det(A) = z^2 - 2z + 5 = (z - 1)^2 + 2^2. \]
The eigenvalues of $A$ are the roots of this polynomial, which are $1 + 2i$ and $1 - 2i$. Because
\[
e^{tA} = e^t \left[ \cos(2t)I + \frac{\sin(2t)}{2} (A - I) \right]
\]
\[
= e^t \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(2t) + \frac{\sin(2t)}{2} \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \right] = e^t \begin{pmatrix} \cos(2t) + \frac{1}{2} \sin(2t) \\ -2 \sin(2t) + \cos(2t) \end{pmatrix},
\]
the solution of the initial-value problem is
\[
x(t) = e^{tA} x' = e^t \begin{pmatrix} \cos(2t) & \frac{1}{2} \sin(2t) \\ -2 \sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^t \begin{pmatrix} \cos(2t) + \frac{1}{2} \sin(2t) \\ -2 \sin(2t) + \cos(2t) \end{pmatrix}.
\]

(8) [6] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 48 liters and the second contains 32 liters. Brine with a salt concentration of 5 grams per liter flows into the first tank at 4 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 6 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 2 liters per hour. At $t = 0$ there are 23 grams of salt in the first tank and 17 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

**Solution.** Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time $t$ minutes. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time $t$ minutes. Because mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time $t$ are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the concentrations of the brine that flows out of these tanks. We have the following picture.

\[
\begin{array}{c}
\text{5 gr/lit} \\
\text{4 lit/hr}
\end{array}
\rightarrow
\begin{array}{c}
V_1(t) \text{ lit} \\
S_1(t) \text{ gr}
\end{array}
\rightarrow
\begin{array}{c}
C_1(t) \text{ gr/lit} \\
7 \text{ lit/hr}
\end{array}
\rightarrow
\begin{array}{c}
V_2(t) \text{ lit} \\
S_2(t) \text{ gr}
\end{array}
\rightarrow
\begin{array}{c}
C_2(t) \text{ gr/lit} \\
2 \text{ lit/hr}
\end{array}
\]

\[
\begin{array}{c}
C_1(t) \text{ gr/lit} \\
3 \text{ lit/hr}
\end{array}
\leftarrow
\begin{array}{c}
V_1(0) = 48 \text{ lit} \\
S_1(0) = 23 \text{ gr}
\end{array}
\leftarrow
\begin{array}{c}
C_2(t) \text{ gr/lit} \\
6 \text{ lit/hr}
\end{array}
\leftarrow
\begin{array}{c}
V_2(0) = 32 \text{ lit} \\
S_2(0) = 17 \text{ gr}
\end{array}
\]

We are asked to write down an initial-value problem that governs $S_1(t)$ and $S_2(t)$.

The rates work out so there will always be $V_1(t) = 48$ liters of brine in the first tank and $V_2(t) = 32 - t$ liters in the second. Then $S_1(t)$ and $S_2(t)$ are governed by the initial-value problem
\[
\frac{dS_1}{dt} = 5 \cdot 4 + \frac{S_2}{32-t} \cdot 6 - \frac{S_1}{48} \cdot 7 - \frac{S_1}{48} \cdot 3, \quad S_1(0) = 23,
\]
\[
\frac{dS_2}{dt} = \frac{S_1}{48} \cdot 7 - \frac{S_2}{32-t} \cdot 6 - \frac{S_2}{32-t} \cdot 2, \quad S_2(0) = 17.
\]
You could leave the answer in the above form. However, it can be simplified to
\[
\frac{dS_1}{dt} = 20 + \frac{6}{32 - t} S_2 - \frac{5}{24} S_1, \quad S_1(0) = 23,
\]
\[
\frac{dS_2}{dt} = \frac{7}{48} S_1 - \frac{8}{32 - t} S_2, \quad S_2(0) = 17.
\]
Notice that the interval of definition for this initial-value problem is \((-\infty, 32)\).

(9) [8] Compute the Laplace transform of \(f(t) = u(t - 5) e^{2t}\) from its definition. (Here \(u\) is the unit step function.)

**Solution.** The definition of Laplace transform gives
\[
\mathcal{L}[f](s) = \lim_{T \to \infty} \int_0^T e^{-st} u(t - 5) e^{2t} dt = \lim_{T \to \infty} \int_5^T e^{-(s-2)t} dt.
\]
When \(s \leq 2\) this limit diverges to \(+\infty\) because in that case we have for every \(T > 5\)
\[
\int_5^T e^{-(s-2)t} dt \geq \int_5^T dt = T - 5,
\]
which clearly diverges to \(+\infty\) as \(T \to \infty\).

When \(s > 2\) we have for every \(T > 5\)
\[
\int_5^T e^{-(s-2)t} dt = -\left.\frac{e^{-(s-2)t}}{s-2}\right|_5^T = -\frac{e^{-(s-2)T}}{s-2} + \frac{e^{-(s-2)5}}{s-2},
\]
whereby
\[
\mathcal{L}[f](s) = \lim_{T \to \infty} \left[ -\frac{e^{-(s-2)T}}{s-2} + \frac{e^{-(s-2)5}}{s-2} \right] = \frac{e^{-(s-2)5}}{s-2}.
\]

(10) [8] Find the inverse Laplace transforms of the function
\[
F(s) = e^{-4s} \frac{4s + 1}{s^2 - 3s - 10}.
\]
You may refer to the table on the last page.

**Solution.** The denominator factors as \((s - 5)(s + 2)\), so the partial fraction identity is
\[
\frac{4s + 1}{s^2 - 3s - 10} = \frac{4s + 1}{(s - 5)(s + 2)} = \frac{3}{s - 5} + \frac{1}{s + 2}.
\]
Referring to the table on the last page, item 1 with \(a = 5\) and \(n = 0\), and with \(a = -2\) and \(n = 0\) shows that
\[
\mathcal{L}[e^{5t}](s) = \frac{1}{s - 5}, \quad \mathcal{L}[e^{-2t}](s) = \frac{1}{s + 2}.
\]
These formulas also can be obtained from item 2 with \(a = 5\) and \(b = 0\), and with \(a = -2\) and \(b = 0\). From these formulas we obtain
\[
\mathcal{L}^{-1}\left[\frac{4s + 1}{s^2 - 3s - 10}\right](t) = 3\mathcal{L}^{-1}\left[\frac{1}{s - 5}\right] + \mathcal{L}^{-1}\left[\frac{1}{s + 2}\right] = 3e^{5t} + e^{-2t}.
\]
Then item 6 with $c = 4$ and $j(t) = 3e^{5t} + e^{-2t}$ shows that

$$
\mathcal{L}^{-1}\left[\frac{4s + 1}{s^2 - 3s - 10}\right](t) = u(t - 4)\mathcal{L}^{-1}\left[\frac{4s + 1}{s^2 - 3s - 10}\right](t - 4)
= u(t - 4)\left(3e^{5(t-4)} + e^{-2(t-4)}\right).
$$

(11) [12] Consider the following MATLAB commands.

```matlab
>> syms t s Y; f = ['tˆ3 + heaviside(t−2)*(8−tˆ3)'];
>> diffeqn = sym('D(D(y))(t)−4*D(y)(t) + 20*y(t) = ' f);
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, {'laplace(y(t),t,s),t,s'), 'y(0)', 'D(y)(0)'});
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) Give the initial-value problem for $y(t)$ that is being solved.

(b) Find the Laplace transform $Y(s)$ of the solution $y(t)$.

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

**Solution (a).** The initial-value problem for $y(t)$ that is being solved is

$$
y'' - 4y' + 20y = f(t) , \quad y(0) = 3 , \quad y'(0) = -5 ,
$$

where the forcing $f(t)$ can be expressed either as

$$
f(t) = \begin{cases} 
  t^3 & \text{for } 0 \leq t < 2 , \\
  8 & \text{for } 2 \leq t ,
\end{cases}
$$

or in terms of the unit step function as $f(t) = t^3 + u(t-2)(8 - t^3)$.

**Solution (b).** The Laplace transform of the initial-value problem is

$$
\mathcal{L}[y''](s) - 4\mathcal{L}[y'](s) + 20\mathcal{L}[y](s) = \mathcal{L}[f](s),
$$

where

$$
\mathcal{L}[y](s) = Y(s) ,
\mathcal{L}[y'](s) = sY(s) - y(0) = sY(s) - 3 ,
\mathcal{L}[y''](s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 3s + 5 .
$$

To compute $\mathcal{L}[f](s)$, we first write $f(t)$ as

$$
f(t) = t^3 + u(t-2)(8 - t^3) = t^3 + u(t-2)j(t-2) ,
$$

where by setting $j(t-2) = 8 - t^3$ we see that

$$
j(t) = 8 - (t + 2)^3 = 8 - (t^3 + 6t^2 + 12t + 8) = -t^3 - 6t^2 - 12t .
$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 3$, with $a = 0$ and $n = 2$, and with $a = 0$ and $n = 1$ shows that

$$
\mathcal{L}[t^3](s) = \frac{6}{s^4} , \quad \mathcal{L}[t^2](s) = \frac{2}{s^3} , \quad \mathcal{L}[t](s) = \frac{1}{s^2} ,
$$
whereby item 6 with \( c = 2 \) and \( j(t) = -t^3 - 6t^2 - 12t \) shows that
\[
\mathcal{L}[u(t-2)j(t-2)](s) = e^{-2s} \mathcal{L}[j](s) \\
= -e^{-2s} \mathcal{L}[t^3 + 6t^2 + 12t](s) = -e^{-2s} \left( \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} \right).
\]

Therefore
\[
\mathcal{L}[f](s) = \mathcal{L}[t^3 + u(t-2)j(t-2)](s) = \frac{6}{s^4} - e^{-2s} \left( \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} \right).
\]

The Laplace transform of the initial-value problem then becomes
\[
(s^2Y(s) - 3s + 5) - 4(sY(s) - 3) + 20Y(s) = \frac{6}{s^4} - e^{-2s} \left( \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} \right),
\]
which becomes
\[
(s^2 - 4s + 20)Y(s) - 3s + 17 = \frac{6}{s^4} - e^{-2s} \left( \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} \right).
\]

Therefore \( Y(s) \) is given by
\[
Y(s) = \frac{1}{s^2 - 4s + 20} \left( 3s - 17 + \frac{6}{s^4} - e^{-2s} \left( \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} \right) \right).
\]

\[\text{A Short Table of Laplace Transforms}\]

\[
\begin{align*}
\mathcal{L}[t^n e^{at}](s) &= \frac{n!}{(s-a)^{n+1}} \quad \text{for } s > a. \\
\mathcal{L}[e^{at} \cos(bt)](s) &= \frac{s-a}{(s-a)^2 + b^2} \quad \text{for } s > a. \\
\mathcal{L}[e^{at} \sin(bt)](s) &= \frac{b}{(s-a)^2 + b^2} \quad \text{for } s > a. \\
\mathcal{L}[t^n j(t)](s) &= (-1)^n J^{(n)}(s) \quad \text{where } J(s) = \mathcal{L}[j(t)](s). \\
\mathcal{L}[e^{at} j(t)](s) &= J(s-a) \quad \text{where } J(s) = \mathcal{L}[j(t)](s). \\
\mathcal{L}[u(t-c)j(t-c)](s) &= e^{-cs} J(s) \quad \text{where } J(s) = \mathcal{L}[j(t)](s) \quad \text{and } u \text{ is the unit step function.}
\end{align*}
\]