Sample Problems for Second In-Class Exam  
Math 246, Fall 2013, Professor David Levermore

(1) Give the interval of definition for the solution of the initial-value problem
\[\frac{d^3x}{dt^3} + \frac{\cos(3t) \, dx}{4 - t \, dt} = \frac{e^{-2t}}{1 + t}, \quad x(2) = x'(2) = x''(2) = 0.\]

(2) Suppose that \(Y_1(t)\) and \(Y_2(t)\) are solutions of the differential equation
\[y'' + 2y' + (1 + t^2)y = 0.\]
Suppose you know that \(W[Y_1, Y_2](0) = 5\). What is \(W[Y_1, Y_2](t)\)?

(3) The function \(Y(t) = t\) is a solution of the differential equation
\[(t^2 + 4)y'' - 2ty' + 2y = 0.\]
Find a general solution of this equation.

(4) Show that the functions \(Y_1(t) = \cos(t)\), \(Y_1(t) = \sin(t)\), and \(Y_3(t) = 1\) are linearly independent.

(5) Let \(L\) be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are
\[-2 + i3, \ -2 - i3, \ i7, \ i7, \ -i7, \ -i7, \ 5, \ 5, \ -3, \ 0, \ 0.\]
(a) Give the order of \(L\).
(b) Give a general real solution of the homogeneous equation \(Ly = 0\).

(6) Give the natural fundamental set of solutions associated with \(t = 0\) for each of the following equations.
(a) \(y'' - 6y' + 9y = 0\).
(b) \(y'' + 4y' + 20y = 0\).

(7) Let \(D = \frac{d}{dt}\). Solve each of the following initial-value problems.
(a) \(D^2y + 4Dy + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0\).
(b) \(D^2y + 9y = 20e^t, \quad y(0) = 0, \quad y'(0) = 0\).

(8) Let \(D = \frac{d}{dt}\). Give a general real solution for each of the following equations.
(a) \(D^2y + 4Dy + 5y = 3\cos(2t)\).
(b) \(D^2y - y = te^t\).
(c) \(D^2y - y = \frac{1}{1 + e^t}\).
(9) Let $D = \frac{d}{dt}$. Consider the equation

$$L y = D^2 y - 6D y + 25 y = e^t .$$

(a) Compute the Green function $g(t)$ associated with $L$.
(b) Use the Green function to express a particular solution $Y_P(t)$ in terms of definite integrals.

(10) The functions $t$ and $t^2$ are solutions of the homogeneous equation

$$t^2 \frac{d^2 y}{dt^2} - 2t \frac{dy}{dt} + 2y = 0 \quad \text{over } t > 0.$$

(You do not have to check that this is true!)
(a) Compute their Wronskian.
(b) Solve the initial-value problem

$$t^2 \frac{d^2 y}{dt^2} - 2t \frac{dy}{dt} + 2y = t^3 e^t, \quad y(1) = y'(1) = 0, \quad \text{over } t > 0.$$

Try to evaluate all definite integrals explicitly.

(11) What answer will be produced by the following MATLAB commands?

```matlab
>> ode1 = 'D2y + 2*Dy + 5*y = 16*exp(t)';
>> dsolve(ode1, 't')
ans =
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(12) The vertical displacement of a mass on a spring is given by

$$h(t) = 4 e^{-t} \cos(7t) - 3 e^{-t} \sin(7t) ,$$

where positive displacements are upward.
(a) Express $h(t)$ in the form $h(t) = Ae^{-t} \cos(\omega t - \delta)$ with $A > 0$ and $0 \leq \delta < 2\pi$, identifying the quasiperiod and phase of the oscillation. (The phase may be expressed in terms of an inverse trig function.)
(b) Sketch the solution over $0 \leq t \leq 2$.

(13) When a mass of 4 grams is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980 \text{ cm/sec}^2$.) At $t = 0$ the mass is displaced 3 cm above its equilibrium position and is released with no initial velocity. It moves in a medium that imparts a drag force of 2 dynes (1 dyne = 1 gram cm/sec$^2$) when the speed of the mass is 4 cm/sec. There are no other forces. (Assume that the spring force is proportional to displacement and that the drag force is proportional to velocity.)
(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$.

(Do NOT solve this initial-value problem, just write it down!)
(b) What is the natural frequency of the spring?
(c) Show that the system is under damped and find its quasifrequency.