Solutions to Sample Problems for the Math 151a Final Exam  
Professor Levermore, Fall 2014

(1) Let \( f(x) = 3^x \) for every \( x \in \mathbb{R} \).
(a) Use Lagrange interpolation to find a polynomial \( p(x) \) of degree at most two that 
agrees with this function at the points \( x_0 = 0, \ x_1 = 1, \) and \( x_2 = 2 \). (Do not 
simplify!)
(b) Find a bound on \( |f(x) - p(x)| \) for each \( x \in [0, 2] \).

Solution (a). The Lagrange interpolation polynomials for the points \( x_0 = 0, \ x_1 = 1, \) 
and \( x_2 = 2 \) are
\[
\ell_0(x) = \frac{(x-1)(x-2)}{(-1)(-2)} = \frac{(x-1)(x-2)}{2},
\]
\[
\ell_1(x) = \frac{(x-0)(x-2)}{(1)(-1)} = x(2-x),
\]
\[
\ell_2(x) = \frac{(x-0)(x-1)}{(2)(1)} = \frac{x(x-1)}{2}.
\]
The Lagrange interpolation formula therefore yields
\[
p(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + f(x_2)\ell_2(x)
\]
\[
= f(0)\frac{(x-1)(x-2)}{2} + f(1)x(2-x) + f(2)\frac{x(x-1)}{2}
\]
\[
= 3^0\frac{(x-1)(x-2)}{2} + 3x(2-x) + 3^2\frac{x(x-1)}{2}
\]
\[
= \frac{(x-1)(x-2)}{2} + 3x(2-x) + 9\frac{x(x-1)}{2}.
\]
\[\square\]

Solution (b). Because \( f \) is thrice differentiable over \( \mathbb{R} \), the Lagrange Interpolation 
Remainder Theorem states that for every \( x \in [0, 2] \) there exists some \( z_x \in (0, 2) \)
\[
f(x) - p(x) = \frac{1}{3!}f'''(z_x)x(x-1)(x-2).
\]
Because
\[
f'(x) = \log(3)3^x,
\]
\[
f''(x) = (\log(3))^23^x,
\]
\[
f'''(x) = (\log(3))^33^x,
\]
and because \( z_x \in (0, 2) \), we see that
\[
|f'''(z_x)| = (\log(3))^33^{z_x} < (\log(3))^33^2 = 9(\log(3))^3.
\]
Therefore we obtain the pointwise error bound
\[
|f(x) - p(x)| < \frac{1}{6}9(\log(3))^3|x(x-1)(x-2)|
\]
\[
= \frac{3}{2}(\log(3))^3|x(x-1)(x-2)|.
\]
\[\square\]
Remark. We can obtain a uniform error bound over $[0, 2]$ by using calculus to find the maximum value of $|x(x - 1)(x - 2)|$ over $[0, 2]$. This maximum value is found to be $\frac{2}{9}\sqrt{3}$, whereby
\[
|f(x) - p(x)| < \frac{3}{2}(\log(3))^3|x(x - 1)(x - 2)| \leq \frac{1}{3}\sqrt{3}(\log(3))^3.
\]
We can obtain a much cruder uniform error bound without calculus by noting that $|x| \leq 2$, $|x - 1| \leq 1$, and $|x - 2| \leq 2$ for every $x \in [0, 2]$, whereby
\[
|f(x) - p(x)| < \frac{3}{2}(\log(3))^3|x(x - 1)(x - 2)| \leq 6(\log(3))^3.
\]
This uniform bound is $6\sqrt{3}$ times larger than the previous one. This kind of crude uniform bound is presented in the book.

(2) Let $f(x) = 3^x$ for every $x \in \mathbb{R}$. Let $p(x)$ be the polynomial of degree at most two that agrees with this function at the points $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$.

(a) Use divided differences to construct $p(x)$.

(b) Use the Neville algorithm to evaluate $p\left(\frac{1}{2}\right)$.

Solution (a). The one-point divided differences are
\[
f[x_0] = 3^0 = 1, \quad f[x_1] = 3^1 = 3, \quad f[x_2] = 3^2 = 9.
\]
The two-point divided differences are
\[
f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{3 - 1}{1} = 2,
\]
\[
f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{9 - 3}{1} = 6.
\]
The three-point divided difference is
\[
f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{6 - 2}{2} = 2.
\]
Therefore the divided-difference table is
\[
\begin{array}{ccc}
  x_i & f[x_i] & f[x_i, x_{i+1}] & f[x_0, x_1, x_2] \\
  0 & 1 & & 2 \\
  1 & 3 & 6 & \\
  2 & 9 & &
\end{array}
\]
The coefficients for the Newton forward difference formula (as read off from the top entry of each column) are 1, 2, and 2, whereby
\[
p(x) = 1 + 2x + 2x(x - 1).
\]
Alternatively, the coefficients for the Newton backward difference formula (as read off from the bottom entry of each column) are 9, 6, and 2, whereby
\[
p(x) = 9 + 6(x - 2) + 2(x - 2)(x - 1)
\]
Either answer is correct. They are just different ways to write the same polynomial — namely, the polynomial $p(x) = 1 + 2x^2$. \qed
Solution (b). The one-point interpolations are
\[ p_0(\frac{1}{2}) = 3^0 = 1, \quad p_1(\frac{1}{2}) = 3^1 = 3, \quad p_2(\frac{1}{2}) = 3^2 = 9. \]
The two-point (linear) interpolations are
\[
\begin{align*}
    p_{01}(\frac{1}{2}) &= \frac{x_1 - \frac{1}{2}}{x_1 - x_0} p_0(\frac{1}{2}) + \frac{\frac{1}{2} - x_0}{x_1 - x_0} p_1(\frac{1}{2}) = \frac{1 - \frac{1}{2}}{1 - 0} + \frac{\frac{1}{2} - 0}{1 - 0} = 2, \\
    p_{12}(\frac{1}{2}) &= \frac{x_2 - \frac{1}{2}}{x_2 - x_1} p_1(\frac{1}{2}) + \frac{\frac{1}{2} - x_1}{x_2 - x_1} p_2(\frac{1}{2}) = \frac{2 - \frac{1}{2}}{2 - 1} + \frac{\frac{1}{2} - 1}{2 - 1} = 0.
\end{align*}
\]
Finally, three-point (quadratic) interpolation is
\[
\begin{align*}
    p_{012}(\frac{1}{2}) &= \frac{x_2 - \frac{1}{2}}{x_2 - x_0} p_{01}(\frac{1}{2}) + \frac{\frac{1}{2} - x_0}{x_2 - x_0} p_{12}(\frac{1}{2}) = \frac{2 - \frac{1}{2}}{2 - 0} + \frac{\frac{1}{2} - 0}{2 - 0} = 3.
\end{align*}
\]
Therefore the Neville algorithm table is
\[
\begin{array}{cccc}
    x_i & p_i(\frac{1}{2}) & p_{i(i+1)}(\frac{1}{2}) & p_{012}(\frac{1}{2}) \\
    0 & 1 & 2 & \frac{3}{2} \\
    1 & 3 & 0 \\
    2 & 9
\end{array}
\]
Hence, \( p(\frac{1}{2}) = p_{012}(\frac{1}{2}) = \frac{3}{2}. \) \( \square \)

(3) Suppose we wish to approximate definite integrals of the form
\[
I(f) = \int_0^1 f(x) \, dx
\]
by a quadrature approximation of the form
\[
Q(f) = f(0) \, w_0 + f(x_1) \, w_1.
\]
Find \( w_0, \, w_1, \) and \( x_1 \) such that \( Q(f) \) has the highest possible degree of precision.

Solution. We can determine \( w_0, \, w_1, \) and \( x_1 \) by requiring that \( Q[1], \, Q[x], \) and \( Q[x^2] \) be exact. This leads to the equations
\[
\begin{align*}
    1 \cdot w_0 + 1 \cdot w_1 &= Q(1) = I(1) = \int_0^1 1 \cdot x \, dx = \frac{1}{2}, \\
    0 \cdot w_0 + x_1 w_1 &= Q(x) = I(x) = \int_0^1 x \cdot x \, dx = \frac{1}{3}, \\
    0 \cdot w_0 + x_1^2 w_1 &= Q(x^2) = I(x^2) = \int_0^1 x^2 \cdot x \, dx = \frac{1}{4}.
\end{align*}
\]
The last two equations imply \( x_1 = \frac{3}{4} \) and \( w_1 = \frac{4}{9}, \) whereby the first equation yields \( w_0 = \frac{1}{18}. \) The insure that \( Q(f) \) has precision at least 2. Because
\[
Q(x^3) = 0 \cdot w_0 + x_1^3 w_1 = (\frac{3}{4})^3 \frac{4}{9} = \frac{3}{16}, \quad I(x^3) = \int_0^1 x^3 \cdot x \, dx = \frac{1}{5},
\]
we see that \( Q(x^3) \neq I(x^3). \) Therefore \( Q(f) \) has precision 2. \( \square \)
(4) Use Gaussian elimination with backward substitution to solve the linear system
\[
\begin{align*}
  x_1 + x_2 + x_4 &= 2, \\
  2x_1 + x_2 - x_3 + x_4 &= 1, \\
  -x_1 + 2x_2 + 3x_3 - x_4 &= 4, \\
  3x_1 - x_2 - x_3 + 2x_2 &= -3. 
\end{align*}
\]

**Solution.** By applying elementary row operations to the augmented matrix we find
\[
\begin{pmatrix}
  1 & 1 & 0 & 1 & 2 \\
  2 & 1 & -1 & 1 & 1 \\
  -1 & 2 & 3 & -1 & 4 \\
  3 & -1 & -1 & 2 & -3 \\
\end{pmatrix}
\sim
\begin{pmatrix}
  1 & 1 & 0 & 1 & 2 \\
  0 & -1 & -1 & -1 & -3 \\
  0 & 3 & 3 & 0 & 6 \\
  0 & -4 & -1 & -1 & -9 \\
\end{pmatrix}
\]  (subtracting multiples of row 1)
\[
\sim
\begin{pmatrix}
  1 & 1 & 0 & 1 & 2 \\
  0 & -1 & -1 & -1 & -3 \\
  0 & 0 & 0 & -3 & -3 \\
  0 & 0 & 3 & 3 & 3 \\
\end{pmatrix}
\]  (subtracting multiples of row 2)
\[
\sim
\begin{pmatrix}
  1 & 1 & 0 & 1 & 2 \\
  0 & -1 & -1 & -1 & -3 \\
  0 & 0 & 3 & 3 & 3 \\
  0 & 0 & 0 & -3 & -3 \\
\end{pmatrix}
\]  (interchanging rows 3 and 4)
\[
\sim
\begin{pmatrix}
  1 & 1 & 0 & 1 & 2 \\
  0 & 1 & 1 & 1 & 3 \\
  0 & 0 & 1 & 1 & 1 \\
  0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]  (normalizing each row)
\[
\sim
\begin{pmatrix}
  1 & 1 & 0 & 0 & 1 \\
  0 & 1 & 1 & 0 & 2 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]  (subtracting row 4 from other rows)
\[
\sim
\begin{pmatrix}
  1 & 1 & 0 & 0 & 1 \\
  0 & 1 & 0 & 2 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]  (subtracting row 3 from row 2)
\[
\sim
\begin{pmatrix}
  1 & 1 & 0 & 0 & 1 \\
  0 & 1 & 0 & 2 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]  (subtracting row 2 from row 1).

Therefore the solution of the system is \( x_1 = -1, \ x_2 = 2, \ x_3 = 0, \) and \( x_4 = 1. \)  \( \square \)

(5) Let
\[
A = \begin{pmatrix}
  2 & -1 & 3 \\
  2 & 0 & 5 \\
  2 & 1 & 6 \\
\end{pmatrix}.
\]
Find the factorization \( A = LU \) where \( L \) is a lower triangular matrix with ones on its diagonal and \( U \) is an upper triangular matrix.

**Solution.** By forward Gaussian elimination we see that
\[
\begin{pmatrix}
2 & -1 & 3 \\
2 & 0 & 5 \\
2 & 1 & 6
\end{pmatrix}
\sim
\begin{pmatrix}
2 & -1 & 3 \\
0 & 1 & 2 \\
0 & 2 & 3
\end{pmatrix}
\sim
\begin{pmatrix}
2 & -1 & 3 \\
0 & 1 & 2 \\
0 & 0 & -1
\end{pmatrix},
\]
where the elementary row operations in the first step were
Row 2 \(- 1 \cdot \text{Row 1} \) \(\mapsto\) Row 2 ,
Row 3 \(- 1 \cdot \text{Row 1} \) \(\mapsto\) Row 3 ,
while the elementary row operation in the second step was
Row 3 \(- 2 \cdot \text{Row 2} \) \(\mapsto\) Row 3 .

The multipliers used above are recorded in
\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 2 & 1
\end{pmatrix}.
\]

Therefore we can read off that \( A \) has the factorization \( A = LU \) where
\[
L = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 2 & 1
\end{pmatrix}, \quad U = \begin{pmatrix}
2 & -1 & 3 \\
0 & 1 & 2 \\
0 & 0 & -1
\end{pmatrix}.
\]

It is easy to check that \( LU = A \). \(\square\)

(6) For a function \( f \) the Newton divided-difference table is
\[
\begin{array}{cccc}
x_i & f[x_i] & f[x_i, x_{i+1}] & f[x_0, x_1, x_2] \\
0 & 0 & 3 & \quad \\
1 & ? & 3 & \\
2 & ? & \quad & \\
\end{array}
\]

(a) Determine the missing entries in the table.
(b) Give the interpolating polynomial \( p(x) \).

**Solution (a).** Because
\[
3 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{f[x_1, x_2] - 3}{2 - 0} = \frac{1}{2}f[x_1, x_2] - \frac{3}{2},
\]
we see that \( f[x_1, x_2] = 9 \). Because
\[
3 = f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f[x_1] - 0}{1 - 0} = f[x_1],
\]
we see that \( f[x_1] = 3 \). Because
\[
9 = f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{f[x_2] - 3}{2 - 1} = f[x_2] - 3,
\]
we see that \( f[x_2] = 12 \). Therefore the table is

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( f[x_i] )</th>
<th>( f[x_i, x_{i+1}] )</th>
<th>( f[x_0, x_1, x_2] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \square \]

**Solution (b).** The interpolating polynomial \( p(x) \) is

\[
p(x) = 0 + 3 \cdot x + 3 \cdot x(x - 1) = 3x^2.
\]

\[ \square \]

**Remark.** It is easier to do part (b) first. Then we can read off that \( f[x_1] = p(1) = 3 \), \( f[x_2] = p(2) = 12 \), and

\[
f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{12 - 3}{2 - 1} = 9.
\]

(7) Let \( f : [a, b] \rightarrow \mathbb{R} \) be infinitely differentiable. Consider the definite integral

\[
I(f) = \int_a^b f(x) \, dx.
\]

Let \( T_h(f) \) and \( M_h(f) \) denote the approximations of \( I(f) \) by the composite trapezoidal and midpoint rules for uniform subintervals of length \( h \).

(a) Give the formulas for \( T_h(f) \) and \( M_h(f) \) and show that

\[
T_{\frac{h}{2}}(f) = \frac{T_h(f) + M_h(f)}{2}.
\]

(b) The Euler-Maclaurin formula states that the trapezoidal rule satisfies an asymptotic relation of the form

\[
T_h(f) = I(f) + \alpha_2 h^2 + \alpha_4 h^4 + O(h^6) \quad \text{as } h \rightarrow 0,
\]

where the \( \alpha_k \) depend on \( f \), but not on \( h \). Use the relation given in part (a) to show that the midpoint rule \( M_h(f) \) satisfies a similar asymptotic relation.

(c) Apply Richardson extrapolation to \( M_h(f) \) and \( M_{3h}(f) \) to obtain a fourth order quadrature scheme.

**Solution (a).** Let \( h = (b - a)/n \). The composite trapezoidal rule for uniform subintervals of length \( h \) is

\[
T_h(f) = \frac{1}{2} f(a) h + \sum_{j=1}^{n-1} f(a + jh) h + \frac{1}{2} f(b) h.
\]

The composite midpoint rule for uniform subintervals of length \( h \) is

\[
M_h(f) = \sum_{j=1}^{n} f(a + (j - \frac{1}{2})h) h.
\]
Therefore
\[
T_2(f) = \frac{1}{2} f(a) \frac{h}{2} + \sum_{k=1}^{2n-1} f(a + k\frac{h}{2}) \frac{h}{2} + \frac{1}{2} f(b) \frac{h}{2}
\]
\[
= \frac{1}{2} f(a) \frac{h}{2} + \sum_{j=1}^{n-1} f(a + jh) \frac{h}{2} + \frac{1}{2} f(b) \frac{h}{2}
\]
\[
+ \sum_{j=1}^{n} f(a + (2j - 1)\frac{h}{2}) \frac{h}{2}
\]
\[
= T_2(f) = \frac{T_h(f) + M_h(f)}{2}.
\]

Solution (b). The relationship of part (a) implies that
\[
M_h(f) = 2T_2(f) - T_h(f).
\]
The Euler-Maclaurin formula gives
\[
T_h(f) = I(f) + \alpha_2 h^2 + \alpha_4 h^4 + O(h^6),
\]
\[
T_2(f) = I(f) + \frac{1}{4} \alpha_2 h^2 + \frac{1}{16} \alpha_4 h^4 + O(h^6).
\]
Therefore
\[
M_h(f) = 2T_2(f) - T_h(f)
\]
\[
= I(f) - \frac{1}{2} \alpha_2 h^2 - \frac{7}{8} \alpha_4 h^4 + O(h^6) \text{ as } h \to 0.
\]

Solution (c). By part (b) we know that
\[
M_h(f) = I(f) + \beta_2 h^2 + O(h^4) \text{ as } h \to 0.
\]
Hence,
\[
M_{3h}(f) = I(f) + 9 \beta_2 h^2 + O(h^4) \text{ as } h \to 0.
\]
Because
\[
\frac{9M_h(f) - M_{3h}(f)}{8} = I(f) + O(h^4) \text{ as } h \to 0,
\]
a fourth order quadrature method is
\[
Q[f] = \frac{9M_h(f) - M_{3h}(f)}{8}.
\]

(8) Consider the monic polynomials
\[
p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2 - \frac{5}{7}.
\]
(a) Show that these are orthogonal with respect to the weight \( x^4 \) over the interval \([-1, 1]\).
(b) Use these polynomials to derive the one- and two-point Gauss quadrature formulas such that
\[
\int_{-1}^{1} f(x) x^4 \, dx \approx \sum_{i=1}^{n} f(x_i) w_i.
\]

**Solution (a).** Because the integrands have odd symmetry, we see that
\[
\int_{-1}^{1} p_0(x) p_1(x) x^4 \, dx = \int_{-1}^{1} x^5 \, dx = 0,
\]
\[
\int_{-1}^{1} p_1(x) p_2(x) x^4 \, dx = \int_{-1}^{1} x^7 - \frac{5}{7} x^5 \, dx = 0.
\]
We also have
\[
\int_{-1}^{1} p_0(x) p_2(x) x^4 \, dx = \int_{-1}^{1} x^6 - \frac{5}{7} x^4 \, dx = \frac{2}{7} - \frac{5}{7} \cdot \frac{2}{5} = 0.
\]
Therefore \( \{p_0(x), p_1(x), p_2(x)\} \) are orthogonal with respect to the weight \( x^4 \) over the interval \([-1, 1]\).

**Solution (b).** The only root of \( p_1(x) \) is 0. Therefore the one-point Gauss quadrature formula takes the form \( Q[f] = f(0) w \). The weight \( w \) can be determined by requiring that \( Q[1] \) be exact. This leads to the equation
\[
1 \cdot w = Q[1] = \int_{-1}^{1} 1 \cdot x^4 \, dx = \frac{2}{5}.
\]
Therefore the one-point Gauss quadrature is \( Q[f] = f(0) \frac{2}{5} \).

The roots of \( p_2(x) \) are \( \pm \sqrt{\frac{5}{7}} \). Therefore the two-point Gauss quadrature formula takes the form
\[
Q[f] = f(-\sqrt{\frac{5}{7}}) w_- + f(\sqrt{\frac{5}{7}}) w_+.
\]
The weights \( w_- \) and \( w_+ \) can be determined by requiring that \( Q[1] \) and \( Q[x] \) be exact. This leads to the equations
\[
1 \cdot w_- + 1 \cdot w_+ = Q[1] = \int_{-1}^{1} 1 \cdot x^4 \, dx = \frac{2}{5},
\]
\[
-\sqrt{\frac{5}{7}} \cdot w_- + \sqrt{\frac{5}{7}} \cdot w_+ = Q[1] = \int_{-1}^{1} x \cdot x^4 \, dx = 0.
\]
The solution of this system is \( w_- = w_+ = \frac{1}{2} \). Therefore the two-point Gauss quadrature is
\[
Q[f] = f(-\sqrt{\frac{5}{7}}) \frac{1}{2} + f(\sqrt{\frac{5}{7}}) \frac{1}{2}.
\]

(9) Consider approximating \( \sqrt{5} \) by using the Newton(-Raphson) method to approximate the positive zero of \( x^2 - 5 = 0 \). Let \( \{p_n\}_{n=0}^{\infty} \) denote the sequence of Newton iterates corresponding to the initial guess \( p_0 = 3 \).

(a) Find \( g(x) \) such that \( p_{n+1} = g(p_n) \).
(b) Prove that \( \{p_n\}_{n=0}^{\infty} \) is a decreasing sequence that lies within the interval \([\sqrt{5}, 3]\).

(c) Prove that
\[
|p_{n+1} - \sqrt{5}| \leq \frac{1}{2\sqrt{5}} |p_n - \sqrt{5}|^2.
\]

(d) Use the above inequality to show that
\[
|p_n - \sqrt{5}| \leq \left( \frac{1}{2\sqrt{5}} \right)^{2^n-1} |3 - \sqrt{5}|^{2^n}.
\]

**Solution (a).** Let \( f(x) = x^2 - 5 \), so that \( f'(x) = 2x \). The Newton iterates are given by \( p_{n+1} = g(p_n) \) where
\[
g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 5}{2x} = \frac{x^2 + 5}{2x}.
\]

**Solution (b).** Because \( f(2) = 2^2 - 5 = -1 \) and \( f(3) = 3^2 - 5 = 4 \), the solution \( \sqrt{5} \in (2, 3) \). Because
\[
f'(x) = 2x > 0 \quad \text{over} \ (2, 3),
\]
\[
f''(x) = 2 > 0 \quad \text{over} \ (2, 3),
\]
we see that \( f \) is increasing and strictly convex over \((2, 3)\). Therefore the Newton iterates \( \{p_n\}_{n=0}^{\infty} \) form a decreasing sequence that lies within the interval \([\sqrt{5}, 3]\).

**Solution (c).** By the definition of \( p_{n+1} \) as the zero of the tangent line approximation at \( p_n \), the fact \( f(\sqrt{5}) = 0 \), and the Lagrange form of the Taylor remainder for the tangent line approximation at \( p_n \), we have
\[
0 = f(p_n) + f'(p_n)(p_{n+1} - p_n),
\]
\[
0 = f(\sqrt{5}) = f(p_n) + f'(p_n)(\sqrt{5} - p_n) + \frac{1}{2} f''(q_n)(\sqrt{5} - p_n)^2,
\]
for some \( q_n \) between \( p_n \) and \( \sqrt{5} \).

Upon subtracting the second equation from the first we see that
\[
0 = f'(p_n)(p_{n+1} - p_n) - \frac{1}{2} f''(q_n)(\sqrt{5} - p_n)^2.
\]
Upon solving for \( p_{n+1} - p_n \) we obtain the general relation
\[
p_{n+1} - p_n = \frac{f''(q_n)}{2f'(p_n)} (p_n - p_n)^2.
\]
But for \( f(x) = x^2 - 5 \) we have \( f'(x) = 2x \) and \( f''(x) = 2 \), so for every \( x \in [\sqrt{5}, 3] \) we have
\[
2\sqrt{5} \leq f'(x), \quad f''(x) = 2.
\]
Therefore because \( p_n \) and \( q_n \) lie in \([\sqrt{5}, 3]\) while \( p_* = \sqrt{5} \), we obtain the bound
\[
|p_{n+1} - \sqrt{5}| = \frac{f''(q_n)}{2f'(p_n)} |p_n - \sqrt{5}|^2 \leq \frac{2}{2 \cdot 2\sqrt{5}} |p_n - \sqrt{5}|^2,
\]
which reduces to the desired bound.
Solution (d). We give a proof by induction. For \( n = 0 \) we have
\[
|p_0 - \sqrt{5}| = \left( \frac{1}{2\sqrt{5}} \right)^{2^0} |3 - \sqrt{5}|^{2^0},
\]
so the inequality of (d) holds for \( n = 0 \) as an equality. Now assume that the inequality of (d) holds for \( n \). By the inequality of (c) and the inequality of (d) for \( n \) we have
\[
|p_{n+1} - \sqrt{5}| \leq \frac{1}{2\sqrt{5}} |p_n - \sqrt{5}|^2 \leq \frac{1}{2\sqrt{5}} \left| \left( \frac{1}{2\sqrt{5}} \right)^{2^n} |3 - \sqrt{5}|^{2^n} \right|^2 = \frac{1}{2\sqrt{5}} \left( \frac{1}{2\sqrt{5}} \right)^{2^{n+1} - 2} |3 - \sqrt{5}|^{2^{n+1}} = \left( \frac{1}{2\sqrt{5}} \right)^{2^{n+1} - 1} |3 - \sqrt{5}|^{2^{n+1}}.
\]
Therefore the inequality of (d) holds for \( n + 1 \). By induction the inequality of (d) holds for every \( n \). \( \Box \)

(10) Let \( f : [a, b] \to \mathbb{R} \) be infinitely differentiable. Derive an \( O(h^3) \) four-point formula to approximate \( f'(x) \) that uses \( f(x-h), f(x), f(x+h), \) and \( f(x+2h) \).

Solution. By Taylor expansion we have
\[
\begin{align*}
f(x+h) &= f(x) + f'(x)h + \frac{1}{2} f''(x)h^2 + \frac{1}{6} f'''(x)h^3 + O(h^4), \\
f(x-h) &= f(x) - f'(x)h + \frac{1}{2} f''(x)h^2 - \frac{1}{6} f'''(x)h^3 + O(h^4), \\
f(x+2h) &= f(x) + 2f'(x)h + 2f''(x)h^2 + \frac{4}{5} f'''(x)h^3 + O(h^4).
\end{align*}
\]
By solving for difference quotients we obtain
\[
\begin{align*}
\frac{f(x+h) - f(x)}{h} &= f'(x) + \frac{1}{2} f''(x)h + \frac{1}{6} f'''(x)h^2 + O(h^3), \\
\frac{f(x) - f(x-h)}{h} &= f'(x) - \frac{1}{2} f''(x)h + \frac{1}{6} f'''(x)h^2 + O(h^3), \\
\frac{f(x+2h) - f(x)}{2h} &= f'(x) + f''(x)h + \frac{2}{3} f'''(x)h^2 + O(h^3).
\end{align*}
\]
We want to find a weighted average of these formulas for which the \( O(h) \) and \( O(h^2) \) terms cancel. This means we seek weights \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) such that
\[
\begin{align*}
\alpha_1 + \alpha_2 &+ \alpha_3 = 1, \\
\alpha_1 - \alpha_2 &+ 2\alpha_3 = 0, \\
\alpha_1 + \alpha_2 &+ 4\alpha_3 = 0.
\end{align*}
\]
The solution of this system is \( \alpha_1 = 1, \alpha_2 = \frac{1}{3}, \) and \( \alpha_3 = -\frac{1}{3}. \) The weighted average of the formulas is
\[
\begin{align*}
\frac{f(x+h) - f(x)}{h} + \frac{1}{3} \frac{f(x) - f(x-h)}{h} - \frac{1}{3} \frac{f(x+2h) - f(x)}{2h} &= f'(x) + O(h^3).
\end{align*}
\]
Therefore a four-point formula that approximates \( f'(x) \) with a \( O(h^3) \) error is
\[
\begin{align*}
f'(x) &= \frac{f(x+h) - f(x)}{h} + \frac{1}{3} \frac{f(x) - f(x-h)}{h} - \frac{1}{3} \frac{f(x+2h) - f(x)}{2h} + O(h^3) \\
&= \frac{6f(x+h) - 2f(x-h) - 2f(x+2h) - 2f(x)}{6h} + O(h^3).
\end{align*}
\]
(11) Let \( f : [a, b] \to \mathbb{R} \) be infinitely differentiable. Derive a three-point formula that uses \( f(x), f(x + h), \) and \( f(x + 2h) \) to approximate \( f'(x) \) with a \( O(h^2) \) error.

**Solution.** By Taylor expansion we have
\[
\begin{align*}
f(x + h) &= f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + O(h^3), \\
f(x + 2h) &= f(x) + 2f'(x)h + 2f''(x)h^2 + O(h^3).
\end{align*}
\]
By solving for difference quotients we obtain
\[
\begin{align*}
\frac{f(x + h) - f(x)}{h} &= f'(x) + \frac{1}{2}f''(x)h + O(h^2), \\
\frac{f(x + 2h) - f(x)}{2h} &= f'(x) + f''(x)h + O(h^2).
\end{align*}
\]
By subtracting the second from twice the first we find
\[
2 \frac{f(x + h) - f(x)}{h} - \frac{f(x + 2h) - f(x)}{2h} = f'(x) + O(h^2).
\]
Therefore a three-point formula that approximates \( f'(x) \) with a \( O(h^2) \) error is
\[
f'(x) = 2 \frac{f(x + h) - f(x)}{h} - \frac{f(x + 2h) - f(x)}{2h} + O(h^2)
\]
\[
= 4 \frac{f(x + h) - 3f(x) - f(x + 2h)}{2h} + O(h^2).
\]

(12) Suppose that we use Romberg quadrature to approximate the definite integral
\[
\int_0^2 \frac{1}{1 + x} \, dx.
\]
The first three rows of the Romberg table are
\[
\begin{array}{cccc}
R_{k,1} & R_{k,2} & R_{k,3} \\
\frac{4}{3} & \frac{7}{6} & ? \\
\frac{67}{60} & ? & ?
\end{array}
\]
(a) Determine the three missing entries in the table above. (You do not have to simplify arithmetic expressions.)
(b) Which of these entries is the basic Simpson rule?
(c) Which of these entries is the composite Simpson rule for two subintervals?

**Solution (a).** The missing entry in the second row is
\[
R_{2,2} = \frac{4R_{2,1} - R_{1,1}}{4 - 1} = \frac{4 \cdot \frac{7}{6} - \frac{4}{3}}{3}.
\]
The first missing entry in the third row is
\[ R_{3,2} = \frac{4R_{3,1} - R_{2,1}}{4 - 1} = \frac{4 \cdot \frac{67}{60} - \frac{7}{6}}{3}. \]
The second missing entry in the third row is
\[ R_{3,3} = \frac{16R_{3,2} - R_{2,2}}{16 - 1} = \frac{16 \cdot \frac{4 \cdot \frac{7}{6} - \frac{4}{3}}{3} - \frac{4 \cdot \frac{67}{60} \cdot \frac{7}{6}}{3} - \frac{4}{3}}{15}. \]

Solution (b). The \( R_{2,2} \) entry is the basic Simpson rule.

Solution (c). The \( R_{3,2} \) entry is the composite Simpson rule for two subintervals.

Remark. The \( R_{3,3} \) entry is the basic Boole rule.

(13) Find the natural cubic spline that interpolates the data \( f(-1) = -2, f(0) = 0, \) and \( f(1) = 2. \)

Solution. The data has odd symmetry with \( f(0) = 0. \) This means that the cubic spline will have the form
\[ s(x) = \begin{cases} \ ax + bx^2 + cx^3 & \text{for } x \in (0,1), \\ ax - bx^2 + cx^3 & \text{for } x \in (-1,0). \end{cases} \]
Then
\[ s''(x) = \begin{cases} 2b + 6cx & \text{for } x \in (0,1), \\ -2b + 6cx & \text{for } x \in (-1,0). \end{cases} \]
The fact \( s''(x) \) is continuous at \( x = 0 \) implies \( b = 0. \) The fact \( s(x) \) is a natural spline implies that \( s''(-1) = s''(1) = 0, \) which implies \( c = 0. \) The fact \( f(1) = 2 \) then implies \( a = 2. \) Therefore the natural cubic spline that interpolates the data is \( s(x) = 2x. \)

(14) Let \( f(x) = e^{3x}. \) Find a cubic polynomial \( p(x) \) such that
\[ p(0) = f(0), \quad p'(0) = f'(0), \quad p''(0) = f''(0), \quad p(1) = f(1). \]

Solution. Let \( p(x) = a + bx + cx^2 + dx^3. \) Because
\[ p'(x) = b + 2cx + 3dx^2, \quad p''(x) = 2c + 6dx, \]
\[ f'(x) = 3e^{3x}, \quad f''(x) = 9e^{3x}, \]
we see that
\[ p(0) = f(0) \quad \implies a = 1, \]
\[ p'(0) = f'(0) \quad \implies b = 3, \]
\[ p''(0) = f''(0) \quad \implies 2c = 9, \]
\[ p(1) = f(1) \quad \implies a + b + c + d = e^3. \]
We thereby find
\[ a = 1, \quad b = 3, \quad c = \frac{9}{2}, \quad d = e^3 - 3 - \frac{9}{2} = e^3 - \frac{17}{2}. \]
Therefore \( p(x) \) is
\[
p(x) = 1 + 3x + \frac{9}{2}x^2 + (e^3 - \frac{17}{2})x^3.
\]

(15) Consider the matrix
\[
A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & a & -1 \\ 0 & -1 & 2 \end{pmatrix}.
\]

(a) Find all \( a \in \mathbb{R} \) for which \( A \) is invertible.
(b) Find all \( a \in \mathbb{R} \) for which \( A \) is strictly diagonally dominant.
(c) Find all \( a \in \mathbb{R} \) for which \( A \) is positive definite.
(d) Find all \( a \in \mathbb{R} \) for which \( A \) has a factorization \( A = LDL^T \) where \( L \) is a lower triangular matrix with ones on its diagonal and \( D \) is a diagonal matrix with nonzero diagonal entries. (You do not have to find the factorization.)

**Solution (a).** A direct calculation shows that
\[
\det(A) = 2 \cdot a \cdot 2 + (-1) \cdot (-1) \cdot 0 + 0 \cdot (-1) \cdot (-1) \\
- 0 \cdot a \cdot 0 - (-1) \cdot (-1) \cdot 2 \cdot 2 \cdot (-1) \cdot (-1) \\
= 4a - 4 = 4(a - 1).
\]
Therefore \( A \) is invertible if and only if \( a \neq 1 \). \( \square \)

**Remark.** Other characterizations of invertibility can be used to get this result. This one is natural because the calculations can be used again to answer parts (c) and (d).

**Solution (b).** The matrix \( A \) is strictly diagonally dominant if and only if
\[
2 > | -1 | = 1, \quad |a| > | -1 | + | -1 | = 2, \quad 2 > | -1 | = 1.
\]
Therefore \( A \) is strictly diagonally dominant if and only if either \( a < -2 \) or \( a > 2 \). \( \square \)

**Solution (c).** The matrix \( A \) is positive definite if and only if
\[
2 > 0, \quad \det\left( \begin{pmatrix} 2 & -1 \\ -1 & a \end{pmatrix} \right) = 2a - 1 > 0, \quad \det(A) = 4(a - 1) > 0.
\]
Therefore \( A \) is positive definite if and only if \( a > 1 \). \( \square \)

**Remark.** Other characterizations of positive definiteness can be used to get this result. This one is natural because the calculations can be used again to answer part (d).

**Solution (d).** The symmetric matrix \( A \) has a factorization \( A = LDL^T \) where \( L \) is a lower triangular matrix with ones on its diagonal and \( D \) is a diagonal matrix with nonzero diagonal entries if and only if \( A \) can be factorized without pivoting. This is the case if and only if all of its leading principal submatrices have a nonzero determinant, which means
\[
2 \neq 0, \quad \det\left( \begin{pmatrix} 2 & -1 \\ -1 & a \end{pmatrix} \right) = 2a - 1 \neq 0, \quad \det(A) = 4(a - 1) \neq 0.
\]
Therefore \( A \) has such a factorization if and only if \( a \neq \frac{1}{2} \) and \( a \neq 1 \). \( \square \)
(16) Consider the linear system
\[
\begin{align*}
  x_1 - x_2 + x_3 &= 0, \\
  12x_2 - x_3 &= 4, \\
  2x_1 + x_2 + x_3 &= 7.
\end{align*}
\]
Find the row interchanges required to solve this system using Gaussian elimination
(a) with partial pivoting,
(b) with scaled partial pivoting.
(Do not use backward substitution to solve the system!)

**Solution (a).** Applying elementary row operations to the augmented matrix of the
given system using partial pivoting we find that
\[
\begin{pmatrix}
  1 & -1 & 1 & 0 \\
  0 & 12 & -1 & 4 \\
  2 & 1 & 1 & 7
\end{pmatrix}
\sim
\begin{pmatrix}
  2 & 1 & 1 & 7 \\
  0 & 12 & -1 & 4 \\
  1 & -1 & 1 & 0
\end{pmatrix}
\]
(interchanging rows 1 and 3)
\[
\sim
\begin{pmatrix}
  2 & 1 & 1 & 7 \\
  0 & 12 & -1 & 4 \\
  0 & -1.5 & 0.5 & -3.5
\end{pmatrix}
\]
(subtracting half row 1 from row 3)
\[
\sim
\begin{pmatrix}
  2 & 1 & 1 & 7 \\
  0 & 12 & -1 & 4 \\
  0 & 0 & -7.5 & 28.5
\end{pmatrix}
\]
(adding 8 times row 2 to row 3)

Therefore the interchange of rows 1 and 3 in step one is the only one required. □

**Solution (b).** For the first column the scoring of the rows is
\[
s_1 = \max_{j=1,2,3} \{|a_{1j}|\} = 1, \quad s_2 = \max_{j=1,2,3} \{|a_{2j}|\} = 12, \quad s_3 = \max_{j=1,2,3} \{|a_{3j}|\} = 2.
\]
Then
\[
\max_{i=1,2,3} \left\{ \frac{|a_{i1}|}{s_i} \right\} = \max\{1, 0, 1\} = 1,
\]
so no interchange is done. Hence, applying elementary row operations to the aug-mented matrix of the given system using scaled partial pivoting we find that
\[
\begin{pmatrix}
  1 & -1 & 1 & 0 \\
  0 & 12 & -1 & 4 \\
  2 & 1 & 1 & 7
\end{pmatrix}
\sim
\begin{pmatrix}
  1 & -1 & 1 & 0 \\
  0 & 12 & -1 & 4 \\
  0 & 3 & -1 & 7
\end{pmatrix}.
\]

For the second column the scoring of the rows is
\[
s_2 = \max_{j=1,2,3} \{|a_{2j}^{(2)}|\} = 12, \quad s_3 = \max_{j=1,2,3} \{|a_{3j}^{(2)}|\} = 3.
\]
Then
\[
\max_{i=2,3} \left\{ \frac{|a_{i1}^{(2)}|}{s_i} \right\} = \max\{1, 1\} = 1.
\]
so no interchange is done. Hence, applying elementary row operations to the foregoing matrix using scaled partial pivoting we find that

$$
\begin{pmatrix}
1 & -1 & 1 & 0 \\
0 & 12 & -1 & 4 \\
0 & 3 & -1 & 7 \\
\end{pmatrix} \sim \begin{pmatrix}
1 & -1 & 1 & 0 \\
0 & 12 & -1 & 4 \\
0 & 0 & -1 & 0 \\
\end{pmatrix}.
$$

Therefore no row interchanges are required. \(\square\)

(17) Let \(f(x) = e^x + x - 3\) for every \(x \in \mathbb{R}\).

(a) Show that \(f\) has a unique zero \(p^*\) that lies in \([0, 1]\).

(b) Let \(p_0 = 1\). Compute the first Newton iterate \(p_1\).

(c) Let \(\{p_n\}_{n=0}^{\infty}\) be the sequence of Newton iterates with \(p_0 = 1\). Why does this sequence lie above \(p^*\).

(d) Let \(p_0 = 2\) and \(p_1 = 1\). Use the secant method to compute \(p_2\).

(e) Let \(\{p_n\}_{n=0}^{\infty}\) be the sequence of approximates generated by the secant method with \(p_0 = 2\) and \(p_1 = 1\). Why does this sequence lie above \(p^*\).

(f) Let \(p_0 = 0\) and \(p_1 = 1\). Use the false-position method to compute \(p_2\).

(g) Let \(\{p_n\}_{n=0}^{\infty}\) be the sequence of approximates generated by the false-position method with \(p_0 = 0\) and \(p_1 = 1\). Why does this sequence lie below \(p^*\).

**Solution.** Because \(f'(x) = e^x + 1\) is positive for every \(x \in \mathbb{R}\), \(f\) is strictly increasing over \(\mathbb{R}\). Because \(f''(x) = e^x\) is positive for every \(x \in \mathbb{R}\), \(f\) is strictly convex over \(\mathbb{R}\). These facts are noted because they are used several times below.

**Solution (a).** Because \(f : [0, 1] \to \mathbb{R}\) is continuous over \([0, 1]\) while \(f(0) = -2\) and \(f(1) = e + 1 - 3 = e - 2 > 0\) have opposite signs, the Intermediate-Value Theorem implies that \(f\) has at least one zero in \((0, 1)\). On the other hand, because \(f\) is strictly increasing over \(\mathbb{R}\), it has at most one zero over \(\mathbb{R}\). Therefore \(f\) has a unique zero that lies in \((0, 1)\). \(\square\)

**Solution (b).** Because \(f(1) = e - 2\) and \(f'(1) = e + 1\) we have

\[
p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{e - 2}{e + 1} = \frac{3}{e + 1}.
\]

\(\square\)

**Solution (c).** Because \(f\) is increasing and convex (concave up) over \([0, 1]\) the sequence \(\{p_n\}_{n=0}^{\infty}\) of Newton iterates converges monotonically to \(p^*\) from above. \(\square\)

**Solution (d).** Because \(f(2) = e^2 - 1\) and \(f(1) = e - 2\) we have

\[
p_2 = p_1 - \frac{p_1 - p_0}{f(p_1) - f(p_0)} f(p_1) = 1 - \frac{1 - 2}{f(1) - f(2)} f(1) = 1 - \frac{1}{e^2 - e + 1} (e - 2) = \frac{e^2 - 2e + 3}{e^2 - e + 1}.
\]

\(\square\)

**Solution (e).** Because \(f\) is increasing and convex (concave up) over \([0, 2]\) the sequence \(\{p_n\}_{n=0}^{\infty}\) of approximates generated by the secant method converges monotonically to \(p^*\) from above. \(\square\)
Solution (f). Because \( f(0) = -2 \) and \( f(1) = e - 2 \) we have
\[
p_2 = p_1 - \frac{p_1 - p_0}{f(p_1) - f(p_0)} f(p_1) = 1 - \frac{1 - 0}{f(1) - f(0)} f(1)
= 1 - \frac{1}{e + 1} (e - 2) = \frac{3}{e + 1}.
\]
\( \square \)

Solution (g). Because \( f \) is increasing and convex (concave up) over \([0, 1]\) the sequence \( \{p_n\}_{n=0}^\infty \) of approximates generated by the false position method converges monotonically to \( p_* \) from below. \( \square \)

(18) Let \( f \) be a continuous function over \([0, 2]\) such that
\[
\begin{array}{cccccccc}
x & 0 & .25 & .5 & .75 & 1. & 1.25 & 1.5 & 1.75 & 2.0 \\
f(x) & 1.39 & 1.11 & .85 & .43 & .12 & -.23 & -.56 & -.97 & -1.34 \\
\end{array}
\]
(a) Prove that \( f \) has at least one zero in \([0, 2]\).
(b) Find \( p_0, p_1, \) and \( p_2 \) of the bisection method. Write your answers in the blank entries of the following table.

\[
\begin{array}{cccc}
n & a_n & b_n & p_n & f(p_n) \\
0 & 0 & 2 & & \\
1 & & XXXX & & \\
2 & & & & \\
\end{array}
\]

(c) How many iterations will achieve an accuracy of \( 10^{-3} \)?

Solution (a). Because we are given that \( f \) is a continuous function over \([0, 2]\), and because the value 0 lies between the values of \( f(0) = 1.39 \) and \( f(2) = -1.34 \), the Intermediate-Value Theorem implies there exists a \( p_* \in [0, 2] \) such \( f(p_*) = 0. \) \( \square \)

Solution (b). The table should be filled in as follows:

\[
\begin{array}{cccccc}
n & a_n & b_n & p_n & f(p_n) \\
0 & 0 & 2 & 1 & .12 \\
1 & 1 & 2 & 1.5 & -.56 \\
2 & 1 & 1.5 & 1.25 & XXXX \\
\end{array}
\]

\( \square \)

Solution (c). The initial interval \([a_0, b_0]\) is \([0, 2]\), so the width of \([a_n, b_n]\) will be
\[
b_n - a_n = \frac{b_0 - a_0}{2^n} = \frac{2}{2^n}.
\]
Because \( p_n \) is the midpoint of \([a_n, b_n]\), it will be within \( 2^{-n} \) of a zero \( p_* \). We will achieve an accuracy of \( 10^{-3} \) when
\[
\frac{1}{2^n} \leq \frac{1}{1000},
\]
which is the same as
\[
1000 \leq 2^n.
\]
Because \(2^{10} = 1024\), we see that \(n = 10\) is the smallest \(n\) that works. \(\square\)

(19) Define \(g(x) = 1/(2 + x)\) for every \(x \in [0, 1]\).

(a) Show that \(g\) has a unique fixed-point in \([0, 1]\).

(b) Use the Contraction Mapping Theorem to prove the convergence of the fixed-point iteration

\[ p_{n+1} = g(p_n), \quad p_0 = 1. \]

(c) How many iterations will achieve an accuracy of \(10^{-6}\)?

**Solution (a).** Because \(g : [0, 1] \to \mathbb{R}\) is differentiable over \([0, 1]\) with

\[ g'(x) = -\frac{1}{(2 + x)^2}, \]

we see that

\[ -\frac{1}{4} \leq g'(x) < 0 \quad \text{for every} \quad x \in [0, 1]. \]

Because \(g\) is strictly decreasing over \([0, 1]\) with \(g(0) = \frac{1}{2}\) and \(g(1) = \frac{1}{3}\), it follows that

\[ g(x) \in \left[\frac{1}{3}, \frac{1}{2}\right] \subset [0, 1] \quad \text{for every} \quad x \in [0, 1]. \]

Moreover, because \(|g'(x)| \leq \frac{1}{4} < 1\) for every \(x \in [0, 1]\), it follows that

\[ |g(x) - g(y)| \leq \frac{1}{4} |x - y| \quad \text{for every} \quad x, y \in [0, 1]. \]

We conclude that \(g : [0, 1] \to [0, 1]\) is a contraction, whereby the Contraction Mapping Theorem implies that it has a unique fixed point \(p_\ast \in [0, 1]\). \(\square\)

**Remark:** Once we show that \(g : [0, 1] \to [0, 1]\) is continuous, the Fixed-Point Theorem implies the existence of at least one fixed point, but does not assert uniqueness. We obtained the uniqueness result above by arguing that \(g : [0, 1] \to [0, 1]\) is a contraction. We could also have done it by arguing that \(f(x) = x - g(x)\) is a strictly increasing function over \([0, 1]\), whereby \(f\) can have at most one zero. However, this is not the best way to approach part (a) because we will need to show that \(g : [0, 1] \to [0, 1]\) is a contraction to do parts (b) and (c).

**Solution (b).** Because \(g : [0, 1] \to [0, 1]\) is a contraction, the Contraction Mapping Theorem also asserts that the fixed-point iteration

\[ p_{n+1} = g(p_n), \quad p_0 = 1. \]

converges to the unique fixed point \(p_\ast\). \(\square\)

**Solution (c).** Because

\[ |p_{n+1} - p_\ast| = |g(p_n) - g(p_\ast)| \leq \frac{1}{4} |p_n - p_\ast|, \]

we obtain the linear convergence error bound

\[ |p_n - p_\ast| \leq \frac{1}{4^n} |p_0 - p_\ast| = \frac{1}{4^n} |1 - p_\ast| \leq \frac{1}{4^n}. \]

We will always achieve an accuracy of \(10^{-6}\) when

\[ \frac{1}{4^n} \leq 10^{-6}, \]

which is the same as

\[ 4^n \geq 10^6 = 1000^2. \]
Because \(2^{10} = 1024\), we see that \(n = 10\) is the smallest \(n\) that works. \(\square\)

**Remark.** The answer could also be expressed as the smallest \(n\) such that
\[
n \geq \frac{\log(10^6)}{\log(4)} = 3 \frac{\log(10)}{\log(2)} = 3 \log_2(10).
\]

(20) Let \(f(x) = 4^x - 6x^2\).

(a) Prove that \(f\) has at least one zero in \([0, 1]\).

(b) Find \(p_0\) and \(p_1\) of the bisection method. Write your answers in the blank entries of the following table.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(a_n)</th>
<th>(b_n)</th>
<th>(p_n)</th>
<th>(f(p_n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>XXXX</td>
</tr>
</tbody>
</table>

(c) For what value of \(n\) will you achieve an accuracy of \(10^{-3}\)?

**Solution (a).** Because \(f\) is continuous over \([0, 1]\), and because the value 0 lies between the values of \(f(0) = 4^0 - 6 \cdot 0^2 = 1\) and \(f(1) = 4^1 - 6 \cdot 1^2 = -2\), the Intermediate-Value Theorem implies there exists a \(p_* \in (0, 1)\) such \(f(p_*) = 0\). \(\square\)

**Solution (b).** Because \(f(.5) = 4^{.5} - 6 \cdot (.5)^2 = 2 - 1.5 = .5\), the table should be filled in as follows:

<table>
<thead>
<tr>
<th>(n)</th>
<th>(a_n)</th>
<th>(b_n)</th>
<th>(p_n)</th>
<th>(f(p_n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.5</td>
<td>1</td>
<td>.75</td>
<td>XXXX</td>
</tr>
</tbody>
</table>

\(\square\)

**Solution (c).** The initial interval \([a_0, b_0]\) is \([0, 1]\), so the width of \([a_n, b_n]\) will be
\[
b_n - a_n = \frac{b_0 - a_0}{2^n} = \frac{1}{2^n}.
\]

Because \(p_n\) is the midpoint of \([a_n, b_n]\), it will be within \(2^{-n-1}\) of a zero \(p_*.\) We will achieve an accuracy of \(10^{-3}\) when
\[
\frac{1}{2^{n+1}} \leq \frac{1}{1000},
\]

which is the same as
\[
1000 \leq 2^{n+1}.
\]

Because \(2^{10} = 1024\), we see that \(n = 9\) is the smallest \(n\) that works. \(\square\)