Solutions to the Sample Problems for the Final Exam
UCLA Math 135, Winter 2015, Professor David Levermore

Every sample problem for the Midterm exam and every problem on the Midterm exam should be considered a sample problem for the Final exam. Between 30% and 50% of the Final exam will be on that material. Solutions to these problems have already been posted. Some Final exam problems will be drawn from the homework assignments. Here are some sample problems related to material covered since the Midterm exam.

(1) Give a sequence of functions \{f_n\}_{n=1}^\infty over \[0,1\] that converges to the function 0 pointwise over \[0,1\] but that does not converge to 0 in mean square over \[0,1\]. Give reasoning why your example works.

Solution. One example is the sequence \{f_n\}_{n=1}^\infty defined by

\[f_n(x) = 2n\sqrt{x}e^{-nx}.\]

This sequence converges to the function 0 pointwise over \[0,1\] because \(f_n(0) = 0\) for every \(n\) while if \(x \in (0,1]\) then \(e^{-x} < 1\), so that

\[\lim_{n \to \infty} f_n(x) = 2\sqrt{x} \lim_{n \to \infty} ne^{-nx} = 0.\]

This sequence does not converge to 0 in mean square over \[0,1\] because

\[\lim_{n \to \infty} \int_0^1 f_n(x)^2 \, dx = \lim_{n \to \infty} \int_0^1 4n^2 x e^{-2nx} \, dx = \lim_{n \to \infty} \int_0^{2n} y e^{-y} \, dy = \int_0^\infty y e^{-y} \, dy = 1.\]

\[\square\]

Remark. There are many such examples. The easiest way to construct one is to first think of a sequence of nonnegative functions \(\{g_n\}_{n=1}^\infty\) over \[0,1\] that converges to the function 0 pointwise over \[0,1\] but that the area under their graphs over \[0,1\] does not converge to 0 — for example, that satisfies

\[\int_0^1 g_n(x) \, dx = 1, \quad \text{or} \quad \lim_{n \to \infty} \int_0^1 g_n(x) \, dx = 1.\]

Here the 1 on the right-hand sides above can be replaced by any positive number. The sequence \(\{f_n\}_{n=1}^\infty\) is constructed from \(\{g_n\}_{n=1}^\infty\) by setting \(f_n(x) = \sqrt{g_n(x)}\). Then the sequence \(\{f_n\}_{n=1}^\infty\) converges to the function 0 pointwise over \[0,1\] because, by the continuity of the square root, for every \(x \in [0,1]\) we have

\[\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \sqrt{g_n(x)} = \sqrt{\lim_{n \to \infty} g_n(x)} = \sqrt{0} = 0.\]

However, the sequence \(\{f_n\}_{n=1}^\infty\) does not converge to 0 in mean square over \[0,1\] because

\[\lim_{n \to \infty} \int_0^1 f_n(x)^2 \, dx = \lim_{n \to \infty} \int_0^1 g_n(x) \, dx = 1.\]

The example used in the solution given above was constructed in this way from the sequence \(g_n(x) = 4n^2 x e^{-2nx}\). Another example can be constructed in this way from the sequence \(g_n(x) = n^2 x^{n-1}(1-x)\).
(2) Give a sequence of functions \( \{f_n\}_{n=1}^{\infty} \) over \([0, 1]\) that converges to the function 0 in mean square over \([0, 1]\) but such that for every \(x \in [0, 1]\) the sequence of real numbers \( \{f_n(x)\}_{n=1}^{\infty} \) does not converge to the number 0. Give reasoning why your example works.

**Solution.** Let \( \{I_n\}_{n=1}^{\infty} \) be any sequence of intervals contained within \([0, 1]\) such that:

- \(|I_n| \to 0\) as \(n \to \infty\);
- for every \(x \in [0, 1]\) we have \(x \in I_n\) frequently.

Here \(|I_n|\) denotes the length of the interval. The second property means that for every \(x \in [0, 1]\) and every \(m \in \mathbb{N}\) there exits an \(n > m\) such that \(x \in I_n\).

One such sequence of intervals is defined as follows. For every positive integer \(n\) there exists a unique nonnegative integer \(k_n\) such that \(2^{k_n} \leq n < 2^{k_n+1}\). Set

\[
I_n = \left[ \frac{n - 2^{k_n}}{2^{k_n}}, \frac{n + 1 - 2^{k_n}}{2^{k_n}} \right].
\]

Clearly \(|I_n| = 1/2^{k_n} \to 0\) as \(n \to \infty\). Moreover, for every \(x \in [0, 1]\) we have \(x \in I_n\) frequently because for every \(m \in \mathbb{N}\) there exists \(k \in \mathbb{N}\) such that \(2^k > m\) and we have \(\bigcup_{n=2^k}^{2^{k+1}-1} I_n = [0, 1]\), whereby for every \(x \in [0, 1]\) there exists \(n \geq 2^k > m\) such that \(x \in I_n\).

Given any sequence of intervals \( \{I_n\}_{n=1}^{\infty} \) with these properties, a sequence of functions \( \{f_n\}_{n=1}^{\infty} \) over \([0, 1]\) can be constructed by setting \(f_n(x) = 1_{I_n}(x)\), where \(1_I\) denotes the indicator function of the interval \(I\) — namely, the function defined by

\[
1_I(x) = \begin{cases} 
1 & \text{if } x \in I, \\
0 & \text{otherwise}.
\end{cases}
\]

The sequence \( \{f_n\}_{n=1}^{\infty} \) converges to 0 in mean square over \([0, 1]\) because

\[
\int_0^1 f_n(x)^2 \, dx = \int_0^1 1_{I_n}(x)^2 \, dx = \int_0^1 1_{I_n}(x) \, dx = |I_n|,
\]

which implies that

\[
\lim_{n \to \infty} \int_0^1 f_n(x)^2 \, dx = \lim_{n \to \infty} |I_n| = 0.
\]

For every \(x \in [0, 1]\) the sequence of real numbers \( \{f_n(x)\}_{n=1}^{\infty} \) does not converge to the number 0 because for every \(x \in [0, 1]\) we have \(f_n(x) = 1_{I_n}(x) = 1\) frequently.

(3) The Fourier series for \(e^x\) over \(x \in [-\pi, \pi]\) has the form

\[
e^x = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right) \quad \text{for } x \in (-\pi, \pi).
\]

(a) Calculate the \(a_n\) and \(b_n\).

(b) Evaluate \(\lim_{n \to \infty} \sum_{m=1}^{n} (-1)^m a_m\).
Solution (a). The formulas for the Fourier coefficients are

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) e^x \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) e^x \, dx. \]

Primitives for these integrands may be computed with two applications of integration by parts. Alternatively, we can compute these primitives as particular solutions of the first-order nonhomogeneous linear differential equations with constant coefficients given by

\[ Dy_1 = \cos(nx) e^x, \quad Dy_2 = \sin(nx) e^x. \]

These equations have characteristic polynomial \( p(z) = z \), which has root \( z = 0 \). Their forcings have degree \( d = 0 \), characteristic \( \mu + i\nu = 1 + in \), and multiplicity \( m = 0 \). They can be solved by either Key Identity Evaluations or Undetermined Coefficients.

Because \( m + d = 0 \) we need only the Key identity, which is \( D(e^{(1+in)x}) = (1 + in) e^{(1+in)x} \).

Because

\[ D\left(\frac{e^{(1+in)x}}{1+in}\right) = e^{(1+in)x} = e^x \cos(nx) + ie^x \sin(nx), \]

we see that primitives of the integrands are

\[ y_1 = \text{Re}\left(\frac{e^{(1+in)x}}{1+in}\right) = e^x \text{Re}\left(\frac{(1-in)e^{inx}}{1+n^2}\right) \]

\[ = \frac{e^x}{1+n^2} \text{Re}\left((1-in)(\cos(nx) + i \sin(nx))\right) \]

\[ = \frac{e^x}{1+n^2} \left(\cos(nx) + n \sin(nx)\right), \]

\[ y_2 = \text{Im}\left(\frac{e^{(1+in)x}}{1+in}\right) = e^x \text{Im}\left(\frac{(1-in)e^{inx}}{1+n^2}\right) \]

\[ = \frac{e^x}{1+n^2} \text{Im}\left((1-in)(\cos(nx) + i \sin(nx))\right) \]

\[ = \frac{e^x}{1+n^2} \left(\sin(nx) - n \cos(nx)\right). \]

Therefore the Fourier coefficients are given by

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) e^x \, dx = \frac{1}{\pi} \left. \frac{e^x}{1+n^2} \left(\cos(nx) + n \sin(nx)\right) \right|_{-\pi}^{\pi} \]

\[ = (-1)^n \frac{1}{\pi} \frac{e^\pi - e^{-\pi}}{1+n^2}, \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) e^x \, dx = \frac{1}{\pi} \left. \frac{e^x}{1+n^2} \left(\sin(nx) - n \cos(nx)\right) \right|_{-\pi}^{\pi} \]

\[ = -(-1)^n \frac{n}{\pi} \frac{e^\pi - e^{-\pi}}{1+n^2}. \]
Solution (b). The $2\pi$-periodic extension of $e^x$ will have a jump discontinuity at $x = \pi + 2\pi m$ for every $m \in \mathbb{Z}$ where its value jumps from $e^\pi$ to $e^{-\pi}$. Moreover, this extension is piecewise continuously differentiable. By the Dirichlet Convergence Theorem the Fourier Series will converge to the average of these values at each jump. In particular, at $x = \pi$ we have

$$\frac{e^\pi + e^{-\pi}}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\pi) + b_n \sin(n\pi)\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (-1)^n a_n .$$

Because

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \, dx = \frac{1}{\pi} e^\pi \bigg|_{-\pi} = \frac{1}{\pi} \left(e^\pi - e^{-\pi}\right) ,$$

we see that

$$\lim_{n \to \infty} \sum_{m=1}^{n} (-1)^m a_m = \sum_{n=1}^{\infty} (-1)^n a_n = \frac{e^\pi + e^{-\pi}}{2} - \frac{e^\pi - e^{-\pi}}{2\pi} .$$

(4) The sine series for $x^2$ over $x \in [0, \pi]$ has the form

$$x^2 = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{for } x \in [0, \pi) .$$

(a) Calculate the $b_n$.

(b) Evaluate $\lim_{n \to \infty} \sum_{m=1}^{n} b_m \sin(m)$.

Solution (a). The function $x^2$ over $[0, \pi)$ has the odd extension $|x|x$ over $(-\pi, \pi)$. The $2\pi$-periodic extension of this odd extension is piecewise continuously differentiable. Its Fourier coefficients are

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x|x \sin(nx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} x^2 \sin(nx) \, dx .$$

A primitive for this integrand may be computed with two applications of integration by parts. Alternatively, we can compute this primitive as a particular solution of the first-order nonhomogeneous linear differential equation with constant coefficients given by

$$Dy = x^2 \sin(nx) .$$

This equation has characteristic polynomial $p(z) = z$, which has root $z = 0$. Its forcing has degree $d = 2$, characteristic $\mu + i\nu = in$, and multiplicity $m = 0$. It can be solved by either Key Identity Evaluations or Undetermined Coefficients.

Because $m = 0$ and $m+d = 2$ we need the Key identity and its first two derivatives with respect to $z$, which are

$$D(e^{zx}) = ze^{zx}, \quad D(x e^{zx}) = xe^{zx} + e^{zx}, \quad D(x^2 e^{zx}) = x^2 e^{zx} + 2xe^{zx} .$$
Evaluating these at \( z = \ln \) gives
\[
\begin{align*}
D(e^{inx}) &= i\ln e^{inx}, \\
D(x e^{inx}) &= in x e^{inx} + e^{inx}, \\
D(x^2 e^{inx}) &= in x^2 e^{inx} + 2 x e^{inx}.
\end{align*}
\]
Multiplying the first relation by \( 1/(in) \) and subtracting from the second yields
\[
D \left( x e^{inx} - \frac{1}{in} e^{inx} \right) = in \ x e^{inx}.
\]
Multiplying this fourth relation by \( 2/(in) \) and subtracting from the third yields
\[
D \left( x^2 e^{inx} - \frac{2}{in} x e^{inx} - \frac{2}{n^2} e^{inx} \right) = in x^2 e^{inx}.
\]
Because \( x^2 \sin(nx) = \text{Im}(x^2 e^{inx}) \), we divide the above relation by \( in \) and take its imaginary part to see that a primitive of the integrand is
\[
y = \text{Im} \left( \frac{1}{in} x^2 e^{inx} + \frac{2}{n^2} x e^{inx} - \frac{2}{in^3} e^{inx} \right) \\
= -\frac{1}{n} x^2 \cos(nx) + \frac{2}{n^2} x \sin(nx) + \frac{2}{n^3} \cos(nx).
\]
Therefore the Fourier coefficients are given by
\[
b_n = \frac{2}{\pi} \int_0^{\pi} x^2 \sin(nx) \, dx \\
= \frac{2}{\pi} \left( -\frac{1}{n} x^2 \cos(nx) + \frac{2}{n^2} x \sin(nx) + \frac{2}{n^3} \cos(nx) \right) \bigg|_0^{\pi} \\
= \frac{2}{\pi} \left( -\frac{(-1)^n}{n} \pi^2 + 2 \frac{(-1)^n}{n^3} - 2 \frac{1}{n^3} \right).
\]

\[\square\]

Solution (b). The function \( x^2 \) over \([0, \pi)\) has the odd extension \(|x| x\) over \((-\pi, \pi)\). The \(2\pi\)-periodic extension of this odd extension is piecewise continuously differentiable. The Dirichlet Theorem implies that its Fourier series converges at every point of continuity of this extension. Because \( x = 1 \) is such a point of continuity, by plugging \( x = 1 \) into the given relation we see that
\[
1^2 = \sum_{n=1}^{\infty} b_n \sin(n).
\]
Therefore
\[
\lim_{n \to \infty} \sum_{m=1}^{n} b_m \sin(m) = 1.
\]
\[\square\]
(5) The cosine series for \( x^2 \) over \( x \in [-\pi, \pi] \) is
\[
x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4 \cos(n\pi)}{n^2}.
\]

Use this fact to evaluate the following sums. Give your reasoning.

(a) \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)

(b) \( \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n)}{n^2} \)

(c) \( \sum_{n=1}^{\infty} \frac{1}{n^4} \)

**Solution (a).** Because \( x^2 \) is even over \([-\pi, \pi]\), its 2\(\pi\)-periodic extension is continuous, with a piecewise continuous derivative. By the Dirichlet Theorem its Fourier series converges pointwise. Hence, by plugging \( x = \pi \) (or \( x = -\pi \)) into the given relation we see that
\[
\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4 \cos(n\pi)}{n^2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}.
\]
Therefore
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \left( \pi^2 - \frac{\pi^2}{3} \right) = \frac{\pi^2}{6}.
\]

**Solution (b).** Because \( x^2 \) is even over \([-\pi, \pi]\), its 2\(\pi\)-periodic extension is continuous, with a piecewise continuous derivative. By the Dirichlet Theorem its Fourier series converges pointwise. Hence, by plugging \( x = 1 \) (or \( x = -1 \)) into the given relation we see that
\[
1^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4 \cos(n)}{n^2}.
\]
Therefore
\[
\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n)}{n^2} = \frac{1}{4} \left( 1 - \frac{\pi^2}{3} \right).
\]

**Solution (c).** Because \( x^2 \) is even over \([-\pi, \pi]\), its 2\(\pi\)-periodic extension is continuous, with a piecewise continuous derivative. We can read off from the given relation that its Fourier coefficients are
\[
a_0 = \frac{2\pi^2}{3}, \quad a_n = (-1)^n \frac{4}{n^2}, \quad b_n = 0, \quad \text{for } n = 1, 2, \ldots.
\]
The Parseval equality states
\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).
\]
When \( f(x) = x^2 \) we have
\[
\int_{-\pi}^{\pi} (x^2)^2 \, dx = \int_{-\pi}^{\pi} x^4 \, dx = 2 \int_{0}^{\pi} x^4 \, dx = 2 \frac{x^5}{5}\bigg|_{0}^{\pi} = \frac{2\pi^5}{5}.
\]
Hence, the Parseval equality becomes
\[
\frac{2\pi^4}{5} = \frac{1}{2} \frac{4\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}.
\]
Therefore
\[
\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{16} \left( \frac{2\pi^4}{5} - \frac{2\pi^4}{9} \right) = \frac{\pi^4}{90}.
\]
\( \square \)

(6) Find the eigenvalues \( \lambda_n \) and eigenfunctions \( y_n(x) \) that solve the eigenvalue problem
\[
y'' + \lambda y = 0 \quad \text{over} \quad [a, b] \quad \text{with} \quad y(a) = y(b) = 0.
\]

Solution. We break the analysis into three cases: \( \lambda > 0 \), \( \lambda = 0 \), and \( \lambda < 0 \).

Let \( \lambda > 0 \). Set \( k = \sqrt{\lambda} \). Then the differential equation becomes \( y'' + k^2 y = 0 \), which has the general solution
\[
y(x) = c_1 \cos(kx) + c_2 \sin(kx).
\]
The boundary conditions then imply that
\[
0 = c_1 \cos(ka) + c_2 \sin(ka), \quad 0 = c_1 \cos(kb) + c_2 \sin(kb).
\]
This linear system has a nontrivial (nonzero) solution if and only if
\[
0 = \det \begin{pmatrix} \cos(ka) & \sin(ka) \\ \cos(kb) & \sin(kb) \end{pmatrix} = \sin(kb) \cos(ka) - \cos(kb) \sin(ka) = \sin(k(b-a)).
\]
This equation is satisfied if and only if \( k(b-a) = n\pi \) for some positive integer \( n \), by which we conclude that eigenvalues are given by
\[
\lambda_n = \left( \frac{n\pi}{b-a} \right)^2 \quad \text{for} \quad n = 1, 2, \ldots.
\]
Nontrivial solutions of the linear system are
\[
c_1 = -c \sin(ka), \quad c_2 = c \cos(ka), \quad \text{for some} \quad c \neq 0,
\]
whereby nontrivial solutions of the differential equation are
\[
y(x) = -c \sin(ka) \cos(kx) + c \cos(ka) \sin(kx) = c \sin(k(x-a)).
\]
Therefore every eigenfunction associated with \( \lambda_n \) is a multiple of
\[
y_n(x) = \sin \left( \frac{n\pi(x-a)}{b-a} \right).
\]

Let \( \lambda = 0 \). Then the differential equation becomes \( y'' = 0 \), which has the general solution \( y(x) = c_1 + c_2 x \). The boundary conditions then imply that
\[
0 = c_1 + c_2 a, \quad 0 = c_1 + c_2 b.
\]
This linear system has a nontrivial (nonzero) solution if and only if
\[ 0 = \det \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix} = b - a . \]

Because \( b > a \) this equation is not satisfied, whereby \( \lambda = 0 \) is not an eigenvalue.

Let \( \lambda < 0 \). Set \( k = \sqrt{-\lambda} \). Then the differential equation becomes \( y'' - k^2 y = 0 \), which has the general solution
\[ y(x) = c_1 e^{-kx} + c_2 e^{kx} . \]

The boundary conditions then imply that
\[ 0 = c_1 e^{-ka} + c_2 e^{ka} , \quad 0 = c_1 e^{-kb} + c_2 e^{kb} . \]

This linear system has a nontrivial (nonzero) solution if and only if
\[ 0 = \det \begin{pmatrix} e^{-ka} & e^{ka} \\ e^{-kb} & e^{kb} \end{pmatrix} = e^{k(b-a)} - e^{-k(b-a)} . \]

Because \( k > 0 \) and \( b > a \), we see that \( e^{k(b-a)} > 1 \) and \( e^{-k(b-a)} < 1 \), whereby \( e^{k(b-a)} - e^{-k(b-a)} > 0 \). Therefore this equation is not satisfied by any \( k > 0 \), which implies that \( \lambda < 0 \) is never an eigenvalue.

Therefore the eigenvalues \( \lambda_n \) and eigenfunctions \( y_n(x) \) that solve the eigenvalue problem are
\[ \lambda_n = \left( \frac{n\pi}{b-a} \right)^2 , \quad y_n(x) = \sin \left( \frac{n\pi x - a}{b-a} \right) , \quad \text{for } n = 1, 2, \cdots . \]

Any nonzero multiple of \( y_n(x) \) is also an eigenfunction. \( \square \)

(7) Find the eigenvalues \( \lambda_n \) and eigenfunctions \( v_n(x) \) that solve the eigenvalue problem
\[ v'' + \lambda v = 0 \quad \text{over } [0, \pi] \quad \text{with} \quad v'(0) = 0 , \quad v(\pi) = 0 . \]

\textbf{Solution.} We break the analysis into three cases: \( \lambda > 0 \), \( \lambda = 0 \), and \( \lambda < 0 \).

Let \( \lambda > 0 \). Set \( k = \sqrt{\lambda} \). Then the differential equation becomes \( v'' + k^2 v = 0 \), which has the general solution
\[ v(x) = c_1 \cos(kx) + c_2 \sin(kx) , \]

which has derivative
\[ v'(x) = -c_1 k \sin(kx) + c_2 k \cos(kx) . \]

The boundary conditions then imply that
\[ 0 = -c_1 k \sin(0) + c_2 k \cos(0) = c_2 k , \quad 0 = c_1 \cos(k\pi) + c_2 \sin(k\pi) . \]

The first equation implies that \( c_2 = 0 \), whereby the second implies that \( c_1 \cos(k\pi) = 0 \). This linear system has a nontrivial (nonzero) solution if and only if \( \cos(k\pi) = 0 \). This condition is satisfied if and only if \( k\pi = n\pi + \frac{1}{2}\pi \) for some nonnegative integer \( n \), by which we conclude that eigenvalues are given by
\[ \lambda_n = \left( n + \frac{1}{2} \right)^2 \quad \text{for } n = 0, 1, \cdots . \]

Moreover, every eigenfunction associated with \( \lambda_n \) is a multiple of
\[ v_n(x) = \cos \left( (n + \frac{1}{2}) x \right) . \]
Let $\lambda = 0$. Then the differential equation becomes $v'' = 0$, which has the general solution $v(x) = c_1 + c_2 x$, whereby $v'(x) = c_2$. The boundary conditions then imply that

$$0 = c_2, \quad 0 = c_1 + c_2 \pi.$$  

The only solution of this system is $c_2 = c_1 = 0$, which implies that $\lambda = 0$ is not an eigenvalue.

Let $\lambda < 0$. Set $k = \sqrt{-\lambda}$. Then the differential equation becomes $v'' - k^2 v = 0$, which has the general solution

$$v(x) = c_1 e^{-kx} + c_2 e^{kx},$$

whereby

$$v'(x) = -c_1 k e^{-kx} + c_2 k e^{kx}.$$  

The boundary conditions then imply that

$$0 = -c_1 k e^{0} + c_2 k e^{0}, \quad 0 = c_1 e^{-k\pi} + c_2 e^{k\pi}.$$  

This linear system has a nontrivial (nonzero) solution if and only if

$$0 = \det \begin{pmatrix} -k & k \\ e^{-k\pi} & e^{k\pi} \end{pmatrix} = -k e^{k\pi} - k e^{-k\pi}.$$  

Because $k > 0$, we see that $e^{k\pi} > 1$ and $e^{-k\pi} < 1$, whereby $-k (e^{k\pi} - e^{-k\pi}) < 0$. Therefore this equation is not satisfied for any $k > 0$, which implies that $\lambda < 0$ is never an eigenvalue.

Therefore the eigenvalues $\lambda_n$ and eigenfunctions $v_n(x)$ that solve the eigenvalue problem are

$$\lambda_n = \left(n + \frac{1}{2}\right)^2, \quad v_n(x) = \cos\left((n + \frac{1}{2})x\right), \quad \text{for } n = 0, 1, \ldots.$$  

Any nonzero multiple of $v_n(x)$ is also an eigenfunction. \hfill \Box

(8) Find the solution $w(t, x)$ to the vibrating string problem

$$w_{tt} - w_{xx} = 0 \quad \text{over } x \in (0, \pi) \quad \text{with}$$

$$w(t, 0) = w(t, \pi) = 0, \quad w(0, x) = x(\pi - x), \quad w_t(0, x) = 0.$$  

The solution can be expressed as an infinite series.

**Solution.** Use separation of variables. Seek solutions of the partial differential equation in the form $w(t, x) = T(t)S(x)$ that also satisfies the homogeneous initial and boundary conditions. The homogeneous initial condition implies that $T'(0) = 0$. The homogeneous boundary conditions imply that $S(0) = S(\pi) = 0$. The partial differential equation implies that

$$T''(t)S(x) - T(t)S''(x) = 0.$$  

By separating variables we see that there is a constant $\lambda$ such that

$$\frac{T''(t)}{T(t)} = \frac{S''(x)}{S(x)} = -\lambda.$$  

We see that $S(x)$ satisfies the eigenvalue problem

$$S''(x) + \lambda S(x) = 0, \quad S(0) = S(\pi) = 0,$$
which by problem 6 has the eigenvalues $\lambda_n$ and eigenfunctions $S_n(x)$ given by

$$\lambda_n = n^2, \quad S_n(x) = \sin(nx), \quad \text{for } n = 1, 2, \cdots.$$ 

For $\lambda = n^2$ the associated $T(t)$ satisfies

$$T''(t) + n^2 T(t) = 0, \quad T'(0) = 0.$$ 

A general solution of the differential equation is

$$T(t) = c_1 \cos(nt) + c_2 \sin(nt).$$ 

Because $T'(0) = 0$ implies that $c_2 = 0$. Therefore we have found solutions of the partial differential equation in the form $\cos(nt) \sin(nx)$ that also satisfy the homogeneous initial and boundary conditions. By taking a general superposition of these we obtain

$$w(t, x) = \sum_{n=1}^{\infty} b_n \cos(nt) \sin(nx).$$ 

By setting $t = 0$ in this relation we see that $b_n$ are the Fourier sine series coefficients of the initial data — specifically,

$$b_n = \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin(nx) \, dx.$$ 

A primitive for this integrand may be computed with two applications of integration by parts. Alternatively, we can compute this primitive as a particular solution of the first-order nonhomogeneous linear differential equation with constant coefficients given by

$$Dy = (\pi x - x^2) \sin(nx).$$ 

This equation has characteristic polynomial $p(z) = z$, which has root $z = 0$. Its forcing has degree $d = 2$, characteristic $\mu + i\nu = in$, and multiplicity $m = 0$. It can be solved by either Key Identity Evaluations or Undetermined Coefficients.

Because $m = 0$ and $m + d = 2$ we need the Key identity and its first two derivatives with respect to $z$, which are

$$D(e^{zx}) = ze^{zx}, \quad D(x e^{zx}) = xe^{zx} + e^{zx}, \quad D(x^2 e^{zx}) = x^2 e^{zx} + 2xe^{zx}.$$ 

Evaluating these at $z = in$ gives

$$D(e^{inx}) = in e^{inx}, \quad D(x e^{inx}) = in x e^{inx} + e^{inx}, \quad D(x^2 e^{inx}) = in x^2 e^{inx} + 2x e^{inx}.$$ 

Multiplying the first relation by $1/(in)$ and subtracting from the second yields

$$D\left(x e^{inx} - \frac{1}{in} e^{inx}\right) = in x e^{inx}.$$ 

Multiplying this fourth relation by $2/(in)$ and subtracting from the third yields

$$D\left(x^2 e^{inx} - \frac{2}{in} x e^{inx} - \frac{2}{n^2} e^{inx}\right) = in x^2 e^{inx}.$$ 

Because $(\pi x - x^2) \sin(nx) = (x^2 - \pi x) \text{Re}(ie^{inx})$, we find a primitive from the last two relations. The details are omitted. □
(9) Find the solution \( h(t, x) \) to the cooling rod problem
\[
\begin{align*}
\frac{\partial h}{\partial t} &= \frac{\partial^2 h}{\partial x^2} - h \\
\frac{\partial h}{\partial x}(t, 0) &= \frac{\partial h}{\partial x}(t, \pi) = 0, \\
h(0, x) &= \sin(x).
\end{align*}
\]
The solution can be expressed as an infinite series.

**Solution.** Use separation of variables. Seek solutions of the partial differential equation in the form \( h(t, x) = T(t)S(x) \) that also satisfies the homogeneous boundary conditions. The homogeneous boundary conditions imply that \( S'(0) = S'(\pi) = 0 \). The partial differential equation being implies that
\[
\frac{d^2}{dt^2} T(t)S(x) = \frac{d^2}{dx^2} S(x)T(t) - T(t)S(x).
\]
By separating variables we see that there is a constant \( \lambda \) such that
\[
\frac{T'(t)}{T(t)} = \frac{S''(x) - S(x)}{S(x)} = -\lambda - 1.
\]
We see that \( S(x) \) satisfies the eigenvalue problem
\[
S''(x) + \lambda S(x) = 0, \quad S'(0) = S'(\pi) = 0,
\]
It is clear that \( \lambda = 0 \) is an eigenvalue with eigenfunction \( S(x) = 1 \). If \( S(x) \) is an eigenfunction that is not constant then its eigenvalue \( \lambda \) will be nonzero. In that case we set \( Y(x) = S'(x) \) and we see that \( Y(x) \) satisfies the eigenvalue problem
\[
Y''(x) + \lambda Y(x) = 0, \quad Y(0) = Y(\pi) = 0,
\]
which by problem 6 has the eigenvalues \( \lambda_n \) and eigenfunctions \( Y_n(x) \) given by
\[
\lambda_n = n^2, \quad Y_n(x) = \sin(nx), \quad \text{for } n = 1, 2, \ldots.
\]
Therefore the eigenvalues \( \lambda_n \) and eigenfunctions \( S_n(x) \) are given by
\[
\lambda_n = n^2, \quad S_n(x) = \cos(nx), \quad \text{for } n = 0, 1, \ldots.
\]
For \( \lambda = n^2 \) the associated \( T(t) \) satisfies
\[
T'(t) + (n^2 + 1)T(t) = 0.
\]
A general solution of this equation is \( T(t) = e^{-(n^2+1)t} \).

Therefore we have found solutions of the partial differential equation in the form \( e^{-(n^2+1)t} \cos(nx) \) that also satisfy the homogeneous boundary conditions. By taking a general superposition of these we obtain
\[
h(t, x) = \frac{1}{2}a_0e^{-t} + \sum_{n=1}^{\infty} a_n e^{-(n^2+1)t} \cos(nx).
\]
By setting \( t = 0 \) in this relation we see that \( a_n \) are the Fourier cosine series coefficients of the initial data. Because the initial condition is \( h(0, x) = \sin(x) \), we have
\[
a_n = \frac{2}{\pi} \int_0^\pi \cos(nx) \sin(x) \, dx \quad \text{for } n = 0, 1, \ldots.
\]
Upon using the trigonometric identity
\[
\cos(nx) \sin(x) = \frac{1}{2} \left( \sin((n+1)x) - \sin((n-1)x) \right),
\]
we see that \( a_1 = 0 \) and that for \( n \neq 1 \)

\[
a_n = \frac{1}{\pi} \int_0^\pi \left( \sin((n+1)x) - \sin((n-1)x) \right) \, dx
\]

\[
= \frac{1}{\pi} \left[ \frac{\sin((n-1)x)}{n-1} - \frac{\sin((n+1)x)}{n+1} \right]_0^\pi
\]

\[
= -\frac{2}{\pi} \frac{1 + (-1)^n}{n^2 - 1}.
\]

Notice \( a_n = 0 \) for odd \( n \). Therefore the solution is

\[
h(t,x) = \frac{2}{\pi} e^{-t} - \frac{4}{\pi} \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} e^{-(n^2+1)t} \cos(nx).
\]

□

**Remark.** The definite integral giving \( a_n \) can also be evaluated with two integrations by parts. Using the trigonometric identity is certainly faster!

(10) Find the solution \( u(x,y) \) to the boundary-value problem

\[
u_{xx} + u_{yy} = 0 \quad \text{over } (0, 4\pi) \times (-\pi, \pi) \text{ with} \]

\[
u(0,y) = \cos(y)^2, \quad u(4\pi,y) = u(x,-\pi) = u(x,\pi) = 0.
\]

The solution can be expressed as an infinite series.

**Solution.** Use separation of variables. Seek solutions of the partial differential equation in the form \( u(x,y) = X(x)Y(y) \) that also satisfies the homogeneous boundary conditions. The homogeneous boundary conditions imply that \( X(4\pi) = 0 \) and \( Y(-\pi) = Y(\pi) = 0 \). The partial differential equation implies that

\[
X''(x)Y(y) + X(x)Y''(y) = 0.
\]

By separating variables we see that there is a constant \( \lambda \) such that

\[
\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda.
\]

We thereby see that \( Y(y) \) satisfies the eigenvalue problem

\[
Y''(y) + \lambda Y(y) = 0, \quad Y(-\pi) = Y(\pi) = 0,
\]

which by problem 6 has the eigenvalues \( \lambda_n \) and eigenfunctions \( Y_n(y) \) given by

\[
\lambda_n = \frac{n^2}{4}, \quad Y_n(y) = \sin\left(\frac{1}{2}n(y+\pi)\right), \quad \text{for } n = 1, 2, \ldots.
\]

For \( \lambda = \frac{1}{4}n^2 \) the associated \( X(x) \) satisfies

\[
X''(x) - \frac{1}{4}n^2 X(x) = 0, \quad X(4\pi) = 0.
\]

A general solution of this differential equation is \( X(x) = c_1e^{-\frac{1}{2}nx} + c_2e^{\frac{1}{2}nx} \). The boundary condition at \( x = 4\pi \) implies that \( 0 = c_1e^{-2\pi n} + c_2e^{2\pi n} \). Setting \( c_1 = \frac{1}{2}e^{2\pi n} \) and \( c_2 = -\frac{1}{2}e^{-2\pi n} \), we obtain

\[
X_n(x) = \sinh\left(\frac{1}{2}n(4\pi - x)\right).
\]
Therefore we have found solutions of the partial differential equation in the form $X_n(x)Y_n(y)$ that also satisfy the homogeneous boundary conditions. By taking a general superposition of these we obtain

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sinh \left( \frac{1}{2} n (4\pi - x) \right) \sin \left( \frac{1}{2} n (y + \pi) \right).$$

By setting $x = 0$ in this relation we see that $B_n$ are related to the Fourier sine series coefficients $b_n$ of the boundary data by

$$b_n = B_n \sinh(2\pi n), \quad b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(y)^2 \sin \left( \frac{1}{2} n (y + \pi) \right) \, dy.$$

By the trigonometric identity $\cos(y)^2 = \frac{1}{2} (1 + \cos(2y))$, we have

$$b_n = \frac{1}{2\pi} \int_{0}^{2\pi} (1 + \cos(2y)) \sin \left( \frac{1}{2} n y \right) \, dy.$$

By using the trigonometric identity

$$\cos(2y) \sin \left( \frac{1}{2} n y \right) = \frac{1}{2} \left( \sin \left( \frac{1}{2} (n+4)y \right) + \sin \left( \frac{1}{2} (n-4)y \right) \right),$$

we see that $b_n = 0$ for even $n$ and that for odd $n$ we have

$$b_n = \frac{1}{4\pi} \int_{0}^{2\pi} 2 \sin \left( \frac{1}{2} n y \right) + \sin \left( \frac{1}{2} (n+4)y \right) + \sin \left( \frac{1}{2} (n-4)y \right) \, dy$$

$$= -\frac{1}{2\pi} \left( \frac{2 \cos \left( \frac{1}{2} n y \right)}{n} + \frac{\cos \left( \frac{1}{2} (n+4)y \right)}{n+4} + \frac{\cos \left( \frac{1}{2} (n-4)y \right)}{n-4} \right) \bigg|_{0}^{2\pi}$$

$$= \frac{2}{\pi} \frac{n^2 - 8}{n^3 - 16n} \left( 1 - (-1)^n \right) = \frac{4}{\pi} \frac{n^2 - 8}{n^3 - 16n}.$$

Finally, collecting the above results we have

$$u(x, y) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{n^2 - 8}{n^3 - 16n} \frac{\sinh \left( \frac{1}{2} n (4\pi - x) \right)}{\sinh(2\pi n)} \sin \left( \frac{1}{2} n (y + \pi) \right).$$

\[\square\]

(11) Find the solution $u(r, \theta)$ to the Dirichlet problem over the disk $r \leq R$ given by

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad u(R, \theta) = \pi^2 - \theta^2 \quad \text{for} \quad \theta \in (-\pi, \pi).$$

The solution can be expressed as an infinite series.

**Solution.** Use separation of variables. Seek solutions of the partial differential equation in the form $u(r, \theta) = v(r)w(\theta)$ where $v(r)$ is bounded as $r \searrow 0$ and $w(\theta)$ is $2\pi$-periodic. The partial differential equation implies that

$$v''(r)w(\theta) + \frac{1}{r} v'(r)w(\theta) + \frac{1}{r^2} v(r)w''(\theta) = 0.$$

By separating variables we see that there is a constant $\lambda$ such that

$$\frac{r^2 v''(r) + r v'(r)}{v(r)} = -\frac{w''(\theta)}{w(\theta)} = \lambda.$$
We see the $w(\theta)$ satisfies the eigenvalue problem

\[ w''(\theta) + \lambda w(\theta) = 0, \quad w(\theta + 2\pi) = w(\theta). \]

We consider the cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

Let $\lambda > 0$. Set $k = \sqrt{\lambda} > 0$. Then a general solution of the differential equation is $w(\theta) = c_1 \cos(k\theta) + c_2 \sin(k\theta)$. This solution is periodic with period $2\pi/k$. This is $2\pi$-periodic if and only if there must exist a positive integer $n$ such that $n2\pi/k = 2\pi$. Therefore an eigenvalue is $\lambda = n^2$ for some positive integer $n$ with eigenfunctions $w(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta)$.

Let $\lambda = 0$. Then a general solution of the differential equation is $w(\theta) = c_1 + c_2 \theta$. This is $2\pi$-periodic if and only if $c_2 = 0$. Therefore an eigenvalue is $\lambda = 0$ with eigenfunctions $w(\theta) = c$.

Let $\lambda < 0$. Set $k = \sqrt{-\lambda} > 0$. Then a general solution of the differential equation is $w(\theta) = c_1 e^{k\theta} + c_2 e^{-k\theta}$. This is not periodic. Therefore there are no negative eigenvalues.

Because $\lambda = n^2$ for some nonnegative integer $n$, we see that $v(r)$ must be a bounded solution of the differential equation

\[ r^2 v''(r) + rv'(r) - n^2 v(r) = 0, \quad \text{over } [0, R]. \]

We consider the cases $n = 0$ and $n > 0$ separately.

If $n = 0$ then the differential equation for $v(r)$ becomes

\[ rv''(r) + v'(r) = 0. \]

If we think of this as a first-order homogeneous linear equation for $v'(r)$ then we find that $v'(r) = c_1/r$, whereby $v(r) = c_1 \log(r) + c_2$. The requirement that this solution be bounded as $r \searrow 0$ implies that $c_1 = 0$.

If $n > 0$ then we seek solutions of the differential equation in the form $v(r) = r^k$, whereby

\[ 0 = r^2 v''(r) + rv'(r) - n^2 v(r) = k(k - 1)r^k + kr^k - n^2 r^k = (k^2 - n^2)r^k. \]

By taking $k = \pm n$ we obtain the solutions $v(r) = r^n$ and $v(r) = r^{-n}$. A general solution has the form $v(r) = c_1 r^n + c_2 r^{-n}$. The requirement that this solution be bounded as $r \searrow 0$ implies that $c_2 = 0$.

Therefore we have found solutions of the partial differential equation in the forms $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$. By taking a general superposition of these we obtain

\[ u(r, \theta) = A_0 + \sum_{n=1}^{\infty} \left( A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) \right). \]

At $r = R$ we have

\[ u(R, \theta) = A_0 + \sum_{n=1}^{\infty} \left( A_n R^n \cos(n\theta) + B_n R^n \sin(n\theta) \right). \]

Therefore, by the boundary condition at $r = R$ we have

\[ A_0 = \frac{1}{2} a_0, \quad A_n R^n = a_n, \quad B_n R^n = b_n, \quad \text{for } n = 1, 2, \ldots. \]
where

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^2 - \theta^2) \cos(n\theta) \, d\theta, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^2 - \theta^2) \sin(n\theta) \, d\theta. \]

Because \((\pi^2 - \theta^2)\) is an even function of \(\theta\) over \((-\pi, \pi)\), we see that

\[ a_n = \frac{2}{\pi} \int_{0}^{\pi} (\pi^2 - \theta^2) \cos(n\theta) \, d\theta, \quad b_n = 0. \]

Moreover, we have

\[ a_0 = \frac{2}{\pi} \int_{0}^{\pi} (\pi^2 - \theta^2) \, d\theta = 2 \left( \frac{\pi^2}{2} - \frac{\pi^4}{12} \right) = \frac{4}{3} \pi^2. \]

\[ a_n = -\frac{2}{\pi} \int_{0}^{\pi} \theta^2 \cos(n\theta) \, d\theta = -(\pi^2 - \frac{\pi^4}{3}) \frac{2}{n^2} \quad \text{for} \ n = 1, 2, \cdots. \]

Here the value of \(a_n\) for \(n = 1, 2, \cdots\), can be read off from the information given in problem 5, which saves us from computing it.

Finally, upon collecting the above results we obtain

\[ u(r, \theta) = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \frac{r^n}{R^n} \cos(n\theta). \]

(12) Let \(p(x) > 0\) be twice continuously differentiable over \([0, 1]\). Let \(q(x)\) be continuously differentiable over \([0, 1]\). Let \(r(x)\) and \(f(x)\) be continuous over over \([0, 1]\). Show that the problem of solving the second-order nonhomogeneous linear differential equation

\[ p(x)y'' + q(x)y' + r(x)y = f(x), \]

reduces to the problem of solving two first-order nonhomogeneous linear differential equations if we know one positive solution \(\mu(x)\) of the second-order homogeneous linear differential equation

\[ (p(x)\mu)'' - (q(x)\mu)' + r(x)\mu = 0. \]

**Solution.** Multiplying the first differential equation by \(\mu(x)\) and the second by \(y\) we obtain

\[ p(x)\mu(x)y'' + q(x)\mu(x)y' + r(x)\mu(x)y = \mu(x)f(x), \]

\[ \left( p(x)\mu(x) \right)''y - \left( q(x)\mu(x) \right)'y + r(x)\mu(x)y = 0. \]

Upon subtracting the bottom equation from the top one we find that

\[ p(x)\mu(x)y'' - \left( p(x)\mu(x) \right)''y + q(x)\mu(x)y' + \left( q(x)\mu(x) \right)'y = \mu(x)f(x). \]

Because

\[ p(x)\mu(x)y'' - \left( p(x)\mu(x) \right)''y = \left( p(x)\mu(x)y' - \left( p(x)\mu(x) \right)'y \right)', \]

\[ q(x)\mu(x)y' + \left( q(x)\mu(x) \right)'y = \left( q(x)\mu(x)y \right)' \]

we see that

\[ \left( p(x)\mu(x)y' - \left( p(x)\mu(x) \right)'y \right)' + \left( q(x)\mu(x)y \right)' = \mu(x)f(x). \]
Hence, $y$ solves the first-order nonhomogeneous linear differential equation

\[ p(x)\mu(x)y' - (p(x)\mu(x))'y + q(x)\mu(x)y = \int \mu(x)f(x) \, dx. \]

Because we have assumed that $p(x)$ and $\mu(x)$ are positive and continuous over $[0, 1]$, this equation can be put into normal form

\[ y' - \left( \frac{p(x)\mu(x)}{p(\mu(x))} \right)'y + \frac{q(x)}{p(x)} \, y = \frac{1}{p(x)\mu(x)} \int \mu(x)f(x) \, dx. \]

Therefore finding an explicit solution reduces to finding three primitives: one to calculate the right-hand side above, and two to solve this nonhomogeneous linear differential equation. \hfill \Box

(13) Let $q(x)$ and $r(x)$ be continuous over over $[a, b]$. Let $y = u(x)$ and $y = v(x)$ be solutions of the boundary-value problem

\[ y'' + q(x)y' + r(x)y = 0, \quad y(a) + y'(a) = 0, \quad y(b) - y'(b) = 0. \]

Show that $u(x)$ and $v(x)$ must be linearly dependent.

**Solution.** Because $u(x)$ and $v(x)$ both satisfy the boundary condition at $x = a$, we see that

\[ u(a) + u'(a) = 0, \quad v(a) + v'(a) = 0. \]

These equations can be put into the matrix form

\[ \begin{pmatrix} u(a) & u'(a) \\ v(a) & v'(a) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]

from which it follows that

\[ 0 = \det \begin{pmatrix} u(a) & u'(a) \\ v(a) & v'(a) \end{pmatrix} = \det \begin{pmatrix} u(a) & v(a) \\ u'(a) & v'(a) \end{pmatrix} = W[u, v](a), \]

where $W[u, v](x)$ denotes the Wronskian of $u$ and $v$. Because $W[u, v](a) = 0$ and because $u$ and $v$ satisfy the same second-order homogeneous linear differential equation over $[a, b]$, the Abel Theorem implies that $W[u, v](x) = 0$ over $[a, b]$. Moreover, because $W[u, v](x) = 0$ over $[a, b]$ and because $u$ and $v$ satisfy the same second-order homogeneous linear differential equation over $[a, b]$, the functions $u(x)$ and $v(x)$ must be linearly dependent. \hfill \Box

**Remark.** The argument can also be begun by using the boundary condition at $x = b$ to infer that $W[u, v](b) = 0$.

(14) Let $R > 1$. Consider the Sturm-Liouville problem

\[ \partial_r[r^2 \partial_r u] + \lambda r^2 u = 0 \quad \text{over} \quad [1, R] \quad \text{with} \quad u(1) = u(R) = 0. \]

(a) Find the eigenvalues $\lambda_n$ and eigenfunctions $u_n(r)$ for this problem. (Hint: Let $v(r) = ru(r)$ and consider the transformed problem for $v(r)$.)

(b) Derive the orthogonality relation satisfies by $u_m(r)$ and $u_n(r)$ when $m \neq n$. 

Solution (a). Let \( u(r) = v(r)/r \). Then
\[
\partial_r u = \frac{\partial_r v}{r} - \frac{v}{r^2},
\]
whereby
\[
0 = \partial_r [r^2 \partial_r u] + \lambda r^2 u = \partial_r [r \partial_r v - v] + \lambda rv
= r \partial_r v + \partial_r v - \partial_r v + \lambda rv = r (\partial_r v + \lambda v).
\]
Therefore \( v \) satisfies the Sturm-Liouville problem
\[
\partial_r r \partial_r v + \lambda v = 0 \quad \text{over } [1, R] \text{ with } \quad v(1) = v(R) = 0.
\]
The Sturm-Liouville problem for \( v \) is the eigenvalue problem for \( y \) of Problem 6 with \( a = 1 \) and \( b = R \). The solution will not be repeated here. The result was that the eigenvalues \( \lambda_n \) and eigenfunctions \( v_n(r) \) are
\[
\lambda_n = \left( \frac{n\pi}{R} \right)^2, \quad v_n(r) = \sin \left( n\pi \frac{r - 1}{R - 1} \right), \quad \text{for } n = 1, 2, \cdots.
\]
Therefore the eigenvalues \( \lambda_n \) and eigenfunctions \( u_n(r) \) for the given Sturm-Liouville problem for \( u \) are
\[
\lambda_n = \left( \frac{n\pi}{R - 1} \right)^2, \quad u_n(r) = \frac{1}{r} \sin \left( n\pi \frac{r - 1}{R - 1} \right), \quad \text{for } n = 1, 2, \cdots.
\]
Any nonzero multiple of \( u_n(r) \) is also an eigenfunction. \( \Box \)

Solution (b). Let \( u_m(r) \) and \( u_n(r) \) be eigenfunctions with \( m \neq n \). Because an \( r^2 \) appears in the \( \lambda \) term of the Sturm-Liouville problem, these eigenfunctions must satisfy the orthogonality relation
\[
\int_1^R u_m(r) u_n(r) r^2 \, dr = 0.
\]
\( \Box \)

Remark. For Sturm-Liouville problems in the general form
\[
\partial_x [p(x) \partial_x u] + \lambda q(x) u + r(x) u = 0 \quad \text{over } [a, b] \text{ with } u(a) = u(b) = 0,
\]
any eigenfunctions \( u_m(x) \) and \( u_n(x) \) that correspond to different eigenvalues satisfy the orthogonality relation
\[
\int_a^b u_m(x) u_n(x) q(x) \, dx = 0.
\]
This is the only fact you need to know to answer part b.

(15) Find a first-order ordinary differential equation satisfied by an extreme point of the functional
\[
F[u] = \int_0^1 \frac{1}{2} u_x^2 - \frac{1}{4} (1 - u^2)^2 \, dx
\]
that also satisfies the boundary conditions \( u(0) = 0 \) and \( u(1) = 1 \).

Solution. Because \( f(u, u_x) \) does not depend upon \( x \), the Euler equation can be integrated once to obtain the first-order equation
\[
u_x \partial_{u_x} f(u, u_x) - f(u, u_x) = c.
\]
For \( f(u, u_x) = \frac{1}{2} u_x^2 - \frac{1}{4} (1-u^2)^2 \) this becomes
\[
u_x \cdot u_x - \frac{1}{2} u_x^2 + \frac{1}{4} (1-u^2)^2 = c,
\]
which reduces to
\[
u_x^2 = 2c - \frac{1}{2} (1-u^2)^2.
\]
\[\square\]

(16) Let \((a, b, c) \in \mathbb{R}^3\) with \(a^2 + b^2 + c^2 \neq 0\). Maximize \(f(x, y, z) = ax + by + cz\) subject to the constraint \(x^2 + y^2 + z^2 = 1\).

**Solution.** By the method of Lagrange multipliers, we seek critical points of
\[
H(x, y, z, \lambda) = ax + by + cz - \lambda \frac{1}{2} (x^2 + y^2 + z^2 - 1).
\]
This leads to the equations
\[
0 = \partial_x H(x, y, z, \lambda) = a - \lambda x ,
0 = \partial_y H(x, y, z, \lambda) = b - \lambda y ,
0 = \partial_z H(x, y, z, \lambda) = c - \lambda z ,
0 = \partial_\lambda H(x, y, z, \lambda) = \frac{1}{2} (x^2 + y^2 + z^2 - 1).
\]
The first three equations imply \(a^2 + b^2 + c^2 = \lambda^2 (x^2 + y^2 + z^2)\). Because we have assumed \(a^2 + b^2 + c^2 \neq 0\), we see that \(\lambda \neq 0\) and that \(x^2 + y^2 + z^2 \neq 0\). Hence, the first three equations yield
\[
x = \frac{a}{\lambda} , \quad y = \frac{b}{\lambda} , \quad z = \frac{c}{\lambda} .
\]
When this result is plugged into the last equation, we obtain
\[
\frac{a^2 + b^2 + c^2}{\lambda^2} - 1 = 0 ,
\]
which yields \(\lambda = \pm \sqrt{a^2 + b^2 + c^2}\). Therefore the critical points of \(H(x, y, z, \lambda)\) are \((x, y, z, \lambda)\) where
\[
x = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}} , \quad y = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}} , \quad z = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}} ,
\]
\[
\lambda = \pm \sqrt{a^2 + b^2 + c^2} .
\]
Therefore the critical points of \(f(x, y, z) = ax + by + cz\) subject to the constraint \(x^2 + y^2 + z^2 = 1\) are
\[
(x, y, z) = \pm \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} .
\]
The corresponding critical values are
\[
f(x, y, z) = \pm \sqrt{a^2 + b^2 + c^2} .
\]
Therefore the maximum of \(f(x, y, z) = ax + by + cz\) subject to the constraint \(x^2 + y^2 + z^2 = 1\) is \(\sqrt{a^2 + b^2 + c^2}\). \[\square\]

**Remark.** The maximizer is the critical point
\[
(x, y, z) = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} .
\]
The minimum of \( f(x, y, z) = ax + by + cz \) subject to the constraint \( x^2 + y^2 + z^2 = 1 \) is \(-\sqrt{a^2 + b^2 + c^2}\) and the minimizer is the critical point 
\[
(x, y, z) = -\frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}.
\]

(17) Let \( u(x) \) be continuously differentiable over \([0, \pi]\) with \( u(0) = u(\pi) = 0\). Minimize 
\[
E[u] = \int_0^\pi u'(x)^2 \, dx
\]
subject to the constraint 
\[
\int_0^\pi u(x)^2 \, dx = \pi.
\]

**Solution.** By the method of Lagrange multipliers, we seek critical points of 
\[
F_\lambda[u] = \int_0^\pi \frac{1}{2}u'(x)^2 \, dx - \lambda \frac{1}{2} \left( \int_0^\pi u(x)^2 \, dx - \pi \right)
\]
\[
= \int_0^\pi \frac{1}{2}u'(x)^2 - \lambda \frac{1}{2}(u(x)^2 - 1) \, dx.
\]
The Euler equations for this problem are
\[
\frac{d}{dx}u' + \lambda u = 0, \quad \int_0^\pi u(x)^2 \, dx = \pi.
\]
Recalling the boundary conditions \( u(0) = u(\pi) = 0 \), this shows that every critical point must be a solution of the eigenvalue problem
\[
u'' + \lambda u = 0, \quad u(0) = u(\pi) = 0.
\]
This is problem 6 with \( a = 0 \) and \( b = \pi \). Its eigenvalues \( \lambda_n \) and eigenfunctions \( u_n(x) \) are given by
\[
\lambda_n = n^2, \quad u_n(x) = \sqrt{2} \sin(nx), \quad \text{for } n = 1, 2, \ldots,
\]
where the \( \sqrt{2} \) is chosen so that
\[
\int_0^\pi u_n(x)^2 \, dx = \int_0^\pi 2 \sin(nx)^2 \, dx = \int_0^\pi 1 - \cos(2nx) \, dx = \pi.
\]
These eigenfunctions and their negatives are thereby the critical points of \( E[u] \) subject to the constraint. A direct calculation shows that
\[
E[\pm u_n] = \int_0^\pi (\pm u_n)'(x)^2 \, dx = \int_0^\pi 2n^2 \cos(nx)^2 \, dx = n^2 \int_0^\pi 1 + \cos(2nx) \, dx = n^2 \pi.
\]
Therefore \( \pi \) is the minimum of \( E[u] \) subject to the constraint, which is attained at the minimizers \( \pm \sqrt{2} \sin(x) \). \( \square \)

**Remark.** Strictly speaking, there are gaps in the above argument. However, it is at the level of rigor we have been using in the course. The answer we arrived at is correct, namely, that \( \pi \) is the minimum of \( E[u] \) subject to the constraint. Moreover, it is also true that each eigenfunction \( \pm u_n \) is a critical point of \( E[u] \) with critical value \( n^2 \pi \). This shows a beautiful relationship between eigenvalue problems and variational problems that generalizes in many directions.
(18) Let \((x(t), y(t))\) be a continuously differentiable curve in \(\mathbb{R}^2\) parametrized by \(t \in [0, 1]\) with \((x(0), y(0)) = (-1, 0)\) and \((x(1), y(1)) = (1, 0)\). Minimize 

\[ F[y] = \int_0^1 y(t)x'(t) \, dt \]

subject to the constraint 

\[ \int_0^1 \sqrt{x'(t)^2 + y'(t)^2} \, dt = \frac{2\pi}{3}. \]

**Solution.** By the method of Lagrange multipliers, we seek critical points of 

\[ H_\lambda[x, y] = \int_0^1 y(t)x'(t) \, dt - \lambda \left( \int_0^1 \sqrt{x'(t)^2 + y'(t)^2} \, dt - \frac{2\pi}{3} \right). \]

The Euler equations associated with \(x(t)\) and \(y(t)\) are 

\[ 0 = \frac{d}{dt} \left( y(t) - \lambda \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \right), \]

\[ 0 = x'(t) - \lambda \frac{d}{dt} \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}. \]

These may each be integrated in \(t\) to obtain 

\[ \frac{y(t) - c_1}{\lambda} = \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \quad \frac{x(t) - c_2}{\lambda} = \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}. \]

Therefore 

\[ \frac{(x(t) - c_2)^2}{\lambda^2} + \frac{(y(t) - c_1)^2}{\lambda^2} = 1. \]

The boundary conditions \((x(0), y(0)) = (-1, 0)\) and \((x(1), y(1)) = (1, 0)\) imply that 

\[ \frac{(1 + c_2)^2}{\lambda^2} + \frac{(c_1)^2}{\lambda^2} = 1, \quad \frac{(1 - c_2)^2}{\lambda^2} + \frac{(c_1)^2}{\lambda^2} = 1, \]

which implies that \(c_2 = 0\) and \(\lambda^2 = (c_1)^2 + 1\). Therefore \((x(t), y(t))\) lies on a circle 

\[ x^2 + (y - c)^2 = 1 + c^2 \quad \text{for some } c \in \mathbb{R}. \]

This is a circle with center at \((0, c)\) and radius \(\sqrt{1 + c^2}\). The solution is an arc on this circle of length \(2\pi/3\) that connects the points \((-1, 0)\) and \((1, 0)\). Because the distance between these points is 2 and the length of the arc is \(2\pi/3\), which is just a bit larger than 2, the center of the circle must lie above the \(x\)-axis — i.e. we have \(c > 0\). The half-angle of the arc will have sine \(1/\sqrt{1 + c^2}\), so \(c\) is determined by setting 

\[ \sqrt{1 + c^2} \sin^{-1}\left( \frac{1}{\sqrt{1 + c^2}} \right) = \frac{\pi}{3}. \]

This has solution \(c = \sqrt{3}\), which implies that the circle has radius 2 and the angle of the arc is \(\pi/3\). A picture then shows that the minimum of \(F[x, y]\) is \(\sqrt{3} - \frac{2}{3}\pi\). \(\Box\)

**Remark.** There will be nothing this hard on the final exam! 

Good Luck!