(1) [11] Find an explicit solution to the following initial-value problem.

\[ v' = \cos(t)v^2, \quad v(0) = -\frac{1}{2}. \]

**Solution.** This equation is separable. Its right-hand side is defined everywhere. Its only stationary point is \( v = 0 \). Its separated differential form is

\[ \frac{1}{v^2} \, dv = \cos(t) \, dt, \]

whereby

\[ \int \frac{1}{v^2} \, dv = \int \cos(t) \, dt. \]

By integrating both sides we find that

\[ -\frac{1}{v} = \sin(t) + c. \]

The initial condition \( v(0) = -\frac{1}{2} \) implies that

\[ -\frac{1}{-\frac{1}{2}} = \sin(0) + c, \]

whereby \( c = 2 \). Hence, the solution is governed implicitly by

\[ -\frac{1}{v} = \sin(t) + 2. \]

This can be solved explicitly to obtain

\[ v = -\frac{1}{2 + \sin(t)}. \]

**Remark.** We see that the *interval of definition* of this solution is \(( -\infty, \infty )\). You were not asked for this, but you might have been.

(2) [11] Find an explicit solution to the following initial-value problem.

\[ w' + 3x^2w = e^{-x^3}, \quad w(0) = 5. \]

**Solution.** This equation is linear. It is already in normal form. Its coefficient \( 3x^2 \) and forcing \( e^{-x^3} \) are both continuous everywhere. Therefore interval of definition for each solution is \(( -\infty, \infty )\). (You were not asked for this, but you might have been.)

An integrating factor is

\[ \exp \left( \int_0^x 3z^2 \, dz \right) = e^{x^3}, \]

so the equation has the integrating factor form

\[ \frac{d}{dx} \left( e^{x^3}w \right) = e^{x^3}e^{-x^3} = 1. \]

By integrating this we find

\[ e^{x^3}w = x + c. \]
The initial condition \( w(0) = 5 \) implies that
\[
e^{0}5 = 0 + c,
\]
whereby \( c = 5 \). Therefore the solution is
\[
w = (x + 5)e^{-x^3}.
\]

(3) [10] Consider the differential equation
\[
\frac{du}{dt} = \frac{u^2 - 4}{(6 + u)^2}.
\]

(a) [8] Sketch its phase-line portrait over the interval \(-8 \leq u \leq 4\). Identify all of its stationary points and classify each as being either stable, unstable, or semistable.

(b) [1] If \( u(0) = -5 \), how does the solution \( u(t) \) behave as \( t \to \infty \)?

(c) [1] If \( u(-6) = 0 \), how does the solution \( u(t) \) behave as \( t \to \infty \)?

**Solution (a).** This equation is autonomous. Its right-hand side is undefined at \( u = -6 \), where the denominator \((6 + u)^2\) vanishes, and is differentiable elsewhere. It has stationary points at \( u = -2 \) and \( u = 2 \), where the numerator \( u^2 - 4 = (u + 2)(u - 2) \) vanishes. A sign analysis shows that the phase-line portrait for this equation is

```
+  +  -  +
->->-> o ->->-> • <-• <- • ->->-> u
```


\(-6\)  \(-2\)  \(2\)  undefined  stable  unstable

**Solution (b).** The phase-line shows that if \( u(0) = -5 \) then \( u(t) \to -2 \) as \( t \to \infty \).

**Solution (c).** The phase-line shows that if \( u(-6) = 0 \) then \( u(t) \to -2 \) as \( t \to \infty \).

(4) [6] Give the interval of definition for the solution of the initial-value problem
\[
t^2 \frac{dv}{dt} + \frac{v}{t^2 - 9} = \frac{e^t}{t^2 - 25}, \quad v(-2) = 4.
\]

(You do not have to solve this equation to answer this question!)

**Solution.** This problem is linear in \( v \). Its normal form is
\[
\frac{dv}{dt} + \frac{1}{(t^2 - 9)t^2} v = \frac{e^t}{(t^2 - 25)t^2}.
\]

The interval of definition can be read off from this normal form as follows.
- Notice that the coefficient \( \frac{1}{(t^2 - 9)t^2} \) is undefined at \( t = -3, t = 3, \) and \( t = 0 \), and is continuous elsewhere.
- Notice that the forcing \( \frac{e^t}{(t^2 - 25)t^2} \) is undefined at \( t = -5, t = 5, \) and \( t = 0 \), and is continuous elsewhere.

Therefore the interval of definition is \((-3, 0)\) because
- the initial time \( t = -2 \) is in \((-3, 0)\),
- both the coefficient and forcing are continuous over \((-3, 0)\),
- either the coefficient or the forcing is not defined at \( t = -3 \) and at \( t = 0 \).
(5) Consider the initial-value problem
\[
\frac{dw}{dx} = -xe^{-2w}, \quad w(0) = w_o \text{ for some } w_o \text{ in } (-\infty, \infty).
\]
The solution satisfies
\[
e^{2w} = e^{2w_o} - x^2.
\]
Gives its interval of definition as a function of \(w_o\). (You do not have to find the explicit solution!)

**Solution.** This equation has a unique solution for \(w\) whenever \(e^{2w_o} - x^2 > 0\) that is given by
\[
w = \frac{1}{2} \log(e^{2w_o} - x^2).
\]
The condition \(e^{2w_o} - x^2 > 0\) is equivalent to \(e^{2w_o} > x^2\), which is then equivalent to \(-e^{w_o} < x < e^{w_o}\). Therefore the interval of definition is \((-e^{w_o}, e^{w_o})\).

**Remark.** We did not need to find an explicit expression for \(w\) to answer this question. We just had to notice that as \(w\) increases over \((-\infty, +\infty)\) the value of \(e^{2w}\) increases over \((0, +\infty)\). Hence, a unique solution for \(w\) will exist whenever \(e^{2w_o} - x^2 > 0\).

(6) Suppose we have used a numerical method to approximate the solution of an initial-value problem over the time interval \([1, 6]\) with 1000 uniform time steps. How many uniform time steps do we need to reduce the global error of our approximation by roughly a factor of \(\frac{1}{36}\) if the method we had used was each of the following?

(a) Explicit Euler method
(b) Runge-midpoint method

**Solution (a).** The explicit Euler method is first order, so its error scales like \(h\). To reduce the error by a factor of \(\frac{1}{36}\), we must reduce \(h\) by a factor of \(\frac{1}{36}\). We must increase the number of time steps by a factor of 36, which means we need 36,000 uniform time steps.

**Solution (b).** The Runge-midpoint method is second order, so its error scales like \(h^2\). To reduce the error by a factor of \(\frac{1}{36}\), we must reduce \(h\) by a factor of \(\frac{1}{36^\frac{1}{2}} = \frac{1}{6}\). We must increase the number of time steps by a factor of 6, which means you need 6,000 uniform time steps.

(7) Find an implicit solution to the following initial-value problem.
\[
(ye^{3xy} + 2x^2) \, dx + (xe^{3xy} + e^y) \, dy = 0, \quad y(1) = 0.
\]

**Solution.** This differential form is exact because
\[
\partial_y(ye^{3xy} + 2x^2) = e^{3xy} + 3xye^{3xy} = \partial_x(xe^{3xy} + e^y) = e^{3xy} + 3xye^{3xy}.
\]
Therefore we can find \(H(x, y)\) such that
\[
\partial_x H(x, y) = ye^{3xy} + 2x^2, \quad \partial_y H(x, y) = xe^{3xy} + e^y.
\]
Integrating the first equation with respect to \(x\) yields
\[
H(x, y) = \frac{1}{3}e^{3xy} + \frac{2}{3}x^3 + h(y),
\]
whereby
\[
\partial_y H(x, y) = xe^{3xy} + h'(y).
\]
Plugging this expression for $\partial_y H(x, y)$ into the second equation gives
\[ xe^{3xy} + h'(y) = xe^{3xy} + e^y, \]
which yields $h'(y) = e^y$. Taking $h(y) = e^y$, an implicit general solution is given by
\[ \frac{1}{3} e^{3xy} + \frac{2}{3} x^3 + e^y = c. \]

The initial condition $y(1) = 0$ implies that
\[ \frac{1}{3} e^0 + \frac{2}{3} 1^3 + e^0 = c, \]
whereby $c = 2$. Therefore an implicit solution of the initial-value problem is
\[ \frac{1}{3} e^{3xy} + \frac{2}{3} x^3 + e^y = 2. \]

(8) [11] Find an implicit general solution to the following.
\[ (4x^2y - 2y^2) \, dx + (x^3 - 2xy) \, dy = 0. \]

**Solution.** This differential form is not exact because
\[ \partial_y(4x^2y - 2y^2) = 4x^2 - 4y \neq \partial_x(x^3 - 2xy) = 3x^2 - 2y. \]

Therefore we seek an integrating factor $\mu$ that satisfies
\[ \partial_y((4x^2y - 2y^2)\mu) = \partial_x((x^3 - 2xy)\mu). \]

Expanding the partial derivatives yields
\[ (4x^2y - 2y^2)\partial_y\mu + (4x^2 - 4y)\mu = (x^3 - 2xy)\partial_x\mu + (3x^2 - 2y)\mu. \]

Grouping the $\mu$ terms together gives
\[ (4x^2y - 2y^2)\partial_y\mu + (x^2 - 2y)\mu = (x^3 - 2xy)\partial_x\mu. \]

If we set $\partial_y\mu = 0$ then this becomes
\[ (x^2 - 2y)\mu = (x^3 - 2xy)\partial_x\mu = (x^2 - 2y)x\partial_x\mu, \]
which reduces to $x\partial_x\mu = \mu$. This yields the integrating factor $\mu = x$.

Because $x$ is an integrating factor, the original differential form multiplied by $x$ is exact. This means that
\[ (4x^3y - 2xy^2) \, dx + (x^4 - 2x^2y) \, dy = 0 \]
is exact.

Therefore we can find $H(x, y)$ such that
\[ \partial_x H(x, y) = 4x^3y - 2xy^2, \quad \partial_y H(x, y) = x^4 - 2x^2y. \]

Integrating the first equation with respect to $x$ yields
\[ H(x, y) = x^4y - x^2y^2 + h(y), \]
whereby
\[ \partial_y H(x, y) = x^4 - 2x^2y + h'(y). \]

Plugging this expression for $\partial_y H(x, y)$ into the second equation gives
\[ x^4 - 2x^2y + h'(y) = x^4 - 2x^2y, \]
which yields $h'(y) = 0$. Taking $h(y) = 0$, we obtain the implicit general solution
\[ x^4y - x^2y^2 = c. \]
(9) [10] Consider the following Matlab function m-file.

```matlab
function [t,y] = solveit(tI, yI, tF, n)
t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = tI; y(1) = yI; h = (tF - tI)/n; hhalf = h/2;
for j = 1:n
    fnow = (t(j))^2 - 3*exp(t(j) + y(j));
    thalf = t(j) + hhalf; yhalf = y(j) + hhalf*fnow;
    fhalf = (thalf)^2 - 3*exp(thalf + yhalf);
    t(j + 1) = t(j) + h; y(j + 1) = y(j) + h*fhalf;
end
```

Suppose that the input values are \( tI = 0, \ yI = 0, \ tF = 10, \) and \( n = 50. \)

(a) [3] What initial-value problem is being approximated numerically?
(b) [1] What is the numerical method being used?
(c) [2] What is the step size?
(d) [4] What will be the output values of \( t(2) \) and \( y(2) \)?

**Solution (a).** The initial-value problem being approximated numerically is

\[
\frac{dy}{dt} = t^2 - 3e^{t+y}, \quad y(0) = 0.
\]

**Remark.** An initial-value problem consists of both a differential equation and an initial condition. Both must be given for full credit.

**Solution (b).** The Runge-midpoint method is being used. (This is clear from the “h*fhalf” in last line of the “for” loop.)

**Solution (c).** Because \( tF = 10, \ tI = 0, \) and \( n = 50, \) the step size is

\[
h = \frac{tF - tI}{n} = \frac{10 - 0}{50} = \frac{1}{5} = .2.
\]

**Remark.** The correct values for \( tF, \ tI, \) and \( n \) had to be plugged in to get full credit.

**Solution (d).** Because \( h = .2, \) we have \( hhalf = .1. \)

Because \( tI = 0 \) and \( yI = 0, \) we have \( t(1) = tI = 0, \) and \( y(1) = yI = 0. \)

Setting \( j = 1 \) inside the “for” loop then yields

\[
\begin{align*}
\text{fnow} &= (t(1))^2 - 3*exp(t(1) + y(1)) = 0^2 - 3e^{0+0} = 0 - 3 = -3, \\
\text{thalf} &= t(1) + hhalf = 0 + .1 = .1, \\
\text{yhalf} &= y(1) + hhalf * fnow = 0 + .1 \cdot (-3) = 0 - .3 = -.3, \\
\text{fhalf} &= \text{thalf}^2 - 3*exp(\text{thalf} + \text{yhalf}) = (.1)^2 - 3e^{1+(-.3)} = (.1)^2 - 3e^{-2}, \\
\text{t(2)} &= t(1) + h = 0 + .2 = .2, \\
\text{y(2)} &= y(1) + h \cdot \text{fhalf} = 0 + .2 ((.1)^2 - 3e^{-2}).
\end{align*}
\]

**Remark.** This expression for \( y(2) \) did not have to be simplified to get full credit.
Consider the following Matlab commands.

```
>> [T, Y] = meshgrid(-5.0:1.0:5.0, -5.0:1.0:5.0);
>> S = T.^2 - Y.^3;
>> L = sqrt(1 + S.^2);
>> quiver(T, Y, 1./L, S./L, 0.5)
>> axis tight, xlabel 't', ylabel 'y'
```

(a) What is the differential equation being studied?
(b) What kind of graph will these Matlab commands produce?

**Solution (a).** The differential equation being studied is

\[
\frac{dy}{dt} = t^2 - y^3.
\]

This can be read off from the second command.

**Solution (b).** These Matlab commands will produce a direction field for the above equation in the rectangle \([-5, 5] \times [-5, 5]\) with an 11 \times 11 grid of arrows. The fact it is producing a direction field is seen from the third and fourth commands. The sizes of the rectangle and the grid can be read off from the first command.

In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population such that it would triple every five weeks. There are 85,000 mosquitoes in the area when a flock of birds arrives that eats 25,000 mosquitoes per week. Write down an initial-value problem that governs \(M(t)\), the population of mosquitoes in the area after the flock of birds arrives.

**Solution.** The population tripling every five weeks corresponds to a growth factor of \(3^{\frac{1}{5}} = (e^{\log(3)})^{\frac{1}{5}} = e^{\frac{1}{5} \log(3)}\), which implies a growth rate of \(\frac{1}{5} \log(3)\). Therefore the initial-value problem that \(M\) satisfies is

\[
\frac{dM}{dt} = \frac{1}{5} \log(3)M - 25,000, \quad M(0) = 85,000.
\]

A tank with an open top has a base of 3 square meters and a height of 2 meters. The tank is initially empty when water begins to pour into it at a rate of 5 liters per minute. The water also drains from the tank through a hole in its bottom at a rate of \(4\sqrt{h}\) liters per minute where \(h(t)\) is the height of the water in the tank in meters. Write down an initial-value problem that governs \(h(t)\). (Recall that 1 m\(^3\) = 1000 lit.)

**Solution.** Let \(V(t)\) be the volume (lit) of water in the tank at time \(t\) minutes. We have the following (optional) picture.

We want to write down an initial-value problem that governs \(h(t)\).
Because the tank has a base with an area of 3 m$^2$, the volume of water in the tank is $3h(t)$ m$^3$. Because 1 m$^3 = 1000$ lit, $V(t) = 1000 \cdot 3h(t) = 3000h(t)$ lit. Because $V(t)$ satisfies
\[
\frac{dV}{dt} = \text{RATE IN} - \text{RATE OUT} = 5 - 4\sqrt{h},
\]
the initial-value problem that governs $h(t)$ is
\[
3000 \frac{dh}{dt} = 5 - 4\sqrt{h}, \quad h(0) = 0.
\]
Each term in the differential equation has units of lit/min.

**Remark.** No points were deducted if the area of the base was assumed to be 9 m$^2$. In that case the initial-value problem that governs $h(t)$ is
\[
9000 \frac{dh}{dt} = 5 - 4\sqrt{h}, \quad h(0) = 0.
\]

**Remark.** The above solution is all that was required for full credit. If we had been asked whether or not the tank will overflow then we could have answered as follows. The differential equation is autonomous. Its right-hand side is defined for $h \geq 0$ and is differentiable for $h > 0$. It has one stationary point at $h = \frac{25}{16}$. Its phase-line portrait for $h > 0$ is
\[
\begin{array}{c}
+ \\
\rightarrow \\
\bullet \quad \text{at} \quad \frac{25}{16}
\end{array}
\quad \rightarrow \quad \begin{array}{c}
- \\
\rightarrow \\
\end{array}
\quad \cdot \\
\leftarrow \\
\end{array}
\]

This portrait shows that if $h(0) = 0$ then $h(t) \rightarrow \frac{25}{16}$ as $t \rightarrow \infty$. Because the height of the tank is 2 and $\frac{25}{16} < 2$, the tank will not overflow.