Second In-Class Exam Solutions
Math 246, Professor David Levermore
Thursday, 22 October 2015

(1) [4] Give the interval of definition for the solution of the initial-value problem

\[ w''' - \frac{5}{t} w' + \frac{\cos(5t)}{2 + t} w = \frac{e^t}{4 - t}; \quad w(2) = w'(2) = w''(2) = 0. \]

Solution. The equation is linear and is already in normal form. The coefficient of \(w'\) is undefined at \(t = 0\) and is continuous elsewhere. The coefficient of \(w\) is undefined at \(t = -2\) and is continuous elsewhere. The forcing is undefined at \(t = 4\) and is continuous elsewhere. Therefore the interval of definition is \((0, 4)\) because:
- the initial time \(t = 2\) is in the interval \((0, 4)\);
- all the coefficients and the forcing are continuous over \((0, 4)\);
- the coefficient of \(w'\) is undefined at \(t = 0\), the left endpoint of \((0, 4)\);
- and the forcing is undefined at \(t = 4\), the right endpoint of \((0, 4)\).

(2) [12] Let \(L\) be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are \(-2 + i5, -2 + i5, -2 - i5, -2 - i5, i4, -i4, 3, 3, 0, 0, 0\).

(a) [2] Give the order of \(L\).
(b) [10] Give a general real solution of the homogeneous equation \(Lv = 0\).

Solution (a). There are 12 roots listed above, so the degree of the characteristic polynomial is 12, whereby the order of \(L\) is also 12.

Solution (b). A general solution is

\[ v(t) = c_1 e^{-2t} \cos(5t) + c_2 e^{-2t} \sin(5t) + c_3 t e^{-2t} \cos(5t) + c_4 t e^{-2t} \sin(5t) + c_5 \cos(4t) + c_6 \sin(4t) + c_7 e^{3t} + c_8 t e^{3t} + c_9 t^2 e^{3t} + c_{10} + c_{11} t + c_{12} t^2. \]

Here the fundamental set of solutions is generated as follows:
- the double conjugate pair \(-2 \pm i5\) yields \(e^{-2t} \cos(5t), e^{-2t} \sin(5t), t e^{-2t} \cos(5t), t e^{-2t} \sin(5t)\);
- the single conjugate pair \(\pm i4\) yields \(\cos(4t)\) and \(\sin(4t)\);
- the triple real root 3 yields \(e^{3t}, t e^{3t},\) and \(t^2 e^{3t}\);
- the triple real root 0 yields 1, \(t\), and \(t^2\).

(3) [4] Suppose that \(X_1(t), X_2(t),\) and \(X_3(t)\) are solutions of the differential equation

\[ x''' - 3x'' - \cos(t)x' + e^t x = 0, \]

Suppose you know that \(W[X_1, X_2, X_3](0) = 5\). What is \(W[X_1, X_2, X_3](t)\)?

Solution. The Abel Theorem states that \(w(t) = W[X_1, X_2, X_3](t)\) satisfies the first-order homogeneous linear equation \(w' - 3w = 0\). It follows that \(w(t) = w(0)e^{3t}\). Because \(w(0) = W[X_1, X_2, X_3](0) = 5\), we obtain \(w(t) = 5e^{3t}\). Therefore

\[ W[X_1, X_2, X_3](t) = 5e^{3t}. \]
(4) [12] The functions \( \cos(3t) \) and \( \sin(3t) \) are a fundamental set of solutions to \( u'' + 9u = 0 \).

(a) [9] Find the solution \( U(t) \) to the general initial-value problem

\[
u'' + 9u = 0, \quad u(0) = u_0, \quad u'(0) = u_1.\]

(b) [3] Find the associated natural fundamental set of solutions to \( u'' + 9u = 0 \).

Solution (a). Because we are given that \( \cos(3t) \) and \( \sin(3t) \) is a fundamental set of solutions to \( u'' + 9u = 0 \), a general solution is \( U(t) = c_1 \cos(3t) + c_2 \sin(3t) \). Because \( U''(t) = -3c_1 \sin(3t) + 3c_2 \cos(3t) \), the initial conditions imply

\[
u_0 = U(0) = c_1, \quad u_1 = U'(0) = 3c_2.\]

We solve these equations to obtain

\[
c_1 = u_0, \quad c_2 = \frac{1}{3}u_1.\]

Therefore the solution to the general initial-value problem is

\[
U(t) = u_0 \cos(3t) + u_1 \frac{1}{3}\sin(3t).\]

Solution (b). We see from the above solution to the general initial-value problem that the associated natural fundamental set of solutions is

\[
N_0(t) = \cos(3t), \quad N_1(t) = \frac{1}{3}\sin(3t).\]

(5) [9] Give a general real solution of the equation

\[
D^2w - 5Dw + 6w = 20 \sin(4t), \quad \text{where} \quad D = \frac{d}{dt}.\]

Solution. This is a nonhomogeneous linear equation with constant coefficients. Its characteristic polynomial is

\[
p(z) = z^2 - 5z + 6 = (z - 2)(z - 3),\]

which has the two simple real roots 2 and 3. Therefore a general solution of the associated homogeneous equation is

\[
w_H(t) = c_1 e^{2t} + c_2 e^{3t}.\]

The forcing \( 20 \sin(4t) \) has characteristic form with degree \( d = 0 \) and characteristic \( \mu + i\nu = i4 \), which is a root of \( p(z) \) of multiplicity \( m = 0 \). Therefore we can use either Key Identity Evaluations or Undetermined Coefficients to find a particular solution \( w_p(t) \). Then a general solution will be given by \( w(t) = w_H(t) + w_p(t) \).

Zero Degree Formula. Because \( d = 0, \mu + i\nu = i4, \) and \( m = 0, \) we may use the zero degree formula with \( m = 0. \) Because \( p(z) = z^2 - 5z + 6, \) we see that \( p(i4) = (i4)^2 - 5 \cdot i4 + 60 = -16 - i20 + 6 = -(10 + i20), \) and that

\[
L\left(\frac{e^{4t}}{p(i4)}\right) = L\left(\frac{e^{4t}}{-(10 + i20)}\right) = e^{4t}.\]
Because \( L(w) = 20 \sin(4t) = 20 \Im(e^{4it}) \), we see that a particular solution is

\[
w_P(t) = -\Im\left(\frac{20e^{4it}}{10 + i20}\right) = -2\Im\left(\frac{e^{4it}}{1 + i2} - i2\right) = -2\Im\left(\frac{(1 - i2)e^{4it}}{1^2 + 2^2}\right)
\]

\[
= -\frac{2}{5}\Im((1 - i2)(\cos(4t) + i \sin(4t))) = -\frac{2}{5}(2\cos(4t) + \sin(4t))
\]

\[
= \frac{4}{5}\cos(2t) - \frac{2}{5}\sin(4t).
\]

Therefore a general solution is

\[
w(t) = c_1e^{2t} + c_2e^{3t} + \frac{4}{5}\cos(2t) - \frac{2}{5}\sin(4t).
\]

**Key Identity Evaluations.** Because \( m = m + d = 0 \) for the forcing \( 20 \sin(4t) \), we need only the Key Identity

\[
L(e^{zt}) = p(z)e^{zt} = (z^2 - 5z + 6)e^{zt}.
\]

By evaluating this at the characteristic \( z = i4 \) we obtain

\[
L(e^{4it}) = ((i4)^2 - 5(i4) + 6)e^{4it} = (-16 - i20 + 6)e^{4it} = -(10 + i20)e^{4it}.
\]

Because \( L(w) = 20 \sin(4t) = 20 \Im(e^{4it}) \), we see that a particular solution is

\[
w_P(t) = -\Im\left(\frac{20e^{4it}}{10 + i20}\right) = -2\Im\left(\frac{e^{4it}}{1 + i2} - i2\right) = -2\Im\left(\frac{(1 - i2)e^{4it}}{1^2 + 2^2}\right)
\]

\[
= -\frac{2}{5}\Im((1 - i2)(\cos(4t) + i \sin(4t))) = -\frac{2}{5}(2\cos(4t) + \sin(4t))
\]

\[
= \frac{4}{5}\cos(2t) - \frac{2}{5}\sin(4t).
\]

Therefore a general solution is

\[
w(t) = c_1e^{2t} + c_2e^{3t} + \frac{4}{5}\cos(2t) - \frac{2}{5}\sin(4t).
\]

**Undetermined Coefficients.** Because \( \mu + i\nu = i4 \) and \( m = m + d = 0 \) for the forcing \( 20 \sin(4t) \), we seek a particular solution in the form

\[
w_P(t) = A \cos(4t) + B \sin(4t).
\]

Because

\[
w_P'(t) = -4A \sin(4t) + 4B \cos(4t), \quad w_P''(t) = -16A \cos(4t) - 16B \sin(4t),
\]

we see that

\[
Lw_P(t) = w_P''(t) - 5w_P'(t) + 6w_P(t)
\]

\[
= \left[-16A \cos(4t) - 16B \sin(4t)\right] - 5\left[-4A \sin(4t) + 4B \cos(4t)\right]
\]

\[
+ 6\left[A \cos(4t) + B \sin(4t)\right]
\]

\[
= (-10A - 20B) \cos(4t) + (20A - 10B) \sin(4t).
\]

By setting \( Lw_P(t) = 20 \sin(4t) \), we see that

\[
-10A - 20B = 0, \quad 20A - 10B = 20.
\]

The first equation implies \( A = -2B \), which when placed into the second equation yields \(-50B = 20\). Hence, \( B = -\frac{2}{5} \) and \( A = \frac{4}{5} \), which gives

\[
w_P(t) = \frac{4}{5}\cos(4t) - \frac{2}{5}\sin(4t).
\]
Therefore a general solution is
\[ w(t) = c_1e^{2t} + c_2e^{3t} + \frac{4}{5}\cos(4t) - \frac{2}{5}\sin(4t). \]

(6) [8] What answer will be produced by the following Matlab commands?
```matlab
g = 'D2y - 8*Dy + 20*y = 20*exp(5*t)';
dsolve(g, 't')
```

Solution. The commands ask Matlab to give a general solution of the equation
\[ D^2y - 8Dy + 20y = 20e^{5t}, \quad \text{where} \quad D = \frac{d}{dt}. \]

This is a nonhomogeneous linear equation with constant coefficients. Its characteristic polynomial is
\[ p(z) = z^2 - 8z + 20 = (z - 4)^2 + 4 = (z - 4)^2 + 2^2, \]
which has the conjugate pair of roots \( 4 \pm i2 \). Therefore a general solution of the associated homogeneous equation is
\[ y_H(t) = c_1e^{4t}\cos(2t) + c_2e^{4t}\sin(2t). \]

The forcing \( 20e^{5t} \) has characteristic form with degree \( d = 0 \) and characteristic \( \mu + i\nu = 5 \), which is a root of \( p(z) \) of multiplicity \( m = 0 \). Therefore we can use either Key Identity Evaluations or Undetermined Coefficients to find a particular solution \( y_P(t) \).

Zero Degree Formula. Because \( d = 0, \mu + i\nu = 5, \) and \( m = 0 \), we may use the zero degree formula with \( m = 0 \). Because \( p(z) = z^2 - 8z + 20 \), we see that \( p(5) = 5^2 - 8 \cdot 5 + 20 = 25 - 40 + 20 = 5 \), and that
\[ L\left(\frac{e^{5t}}{p(5)}\right) = L\left(\frac{e^{5t}}{5}\right) = e^{5t}. \]

Because \( L(y) = 20e^{5t} \), we see that a particular solution is \( y_P(t) = 4e^{5t} \). Therefore a general solution is
\[ y(t) = c_1e^{4t}\cos(2t) + c_2e^{4t}\sin(2t) + 4e^{5t}. \]

Remark. Had you forgotten the zero degree formula then you could have derived it by Key Identity Evaluations as in the following solution.

Key Identity Evaluations. Because \( m = m + d = 0 \) we only need the Key identity, which is
\[ L(e^{zt}) = p(z)e^{zt} = (z^2 - 8z + 20)e^{zt}. \]

By evaluating this at the characteristic \( z = 5 \) we obtain
\[ L(e^{5t}) = (5^2 - 8 \cdot 5 + 20)e^{5t} = (25 - 40 + 20)e^{5t} = 5e^{5t}. \]

Because \( L(y) = 20e^{5t} \), we see that a particular solution is \( y_P(t) = 4e^{5t} \). Therefore a general solution is
\[ y(t) = c_1e^{4t}\cos(2t) + c_2e^{4t}\sin(2t) + 4e^{5t}. \]
Undetermined Coefficients. Because $\mu + i\nu = 5$ and $m = m + d = 0$ for the forcing $20e^{5t}$, we seek a particular solution in the form 

$$y_P(t) = Ae^{5t}.$$ 

Because $y_P'(t) = 5Ae^{5t}$ and $y_P''(t) = 25Ae^{5t}$, we see that

$$L y_P(t) = y_P''(t) - 8y_P'(t) + 20y_P(t) = [25Ae^{5t}] - 8[5Ae^{5t}] + 20[Ae^{5t}] = (25 - 40 + 20)Ae^{5t} = 5Ae^{5t}.$$ 

By setting $L y_P(t) = 20e^{5t}$, we see that $5A = 20$, whereby $A = 4$. Hence, we obtain the particular solution $y_P(t) = 4e^{5t}$. Therefore a general solution is

$$y(t) = c_1e^{4t}\cos(2t) + c_2e^{4t}\sin(2t) + 4e^{5t}.$$

(7) [8] Compute the Green function $g(t)$ associated with the differential operator

$$D^2 + 8D + 16,$$ 

where $D = \frac{d}{dt}$.

Solution. Because the differential operator has constant coefficients, the Green function $g(t)$ associated with it satisfies the initial-value problem

$$D^2g + 8Dg + 16g = 0, \quad g(0) = 0, \quad g'(0) = 1.$$ 

The characteristic polynomial is

$$p(z) = z^2 + 8z + 16 = (z + 4)^2,$$ 

which has the double real root $-4$. Hence, the Green function has the form

$$g(t) = c_1e^{-4t} + c_2te^{-4t}.$$ 

The initial condition $g(0) = 0$ implies that $c_1 = 0$. Because

$$g'(t) = -4c_2te^{-4t} + c_2e^{-4t},$$ 

the initial condition $g'(0) = 1$ implies that $c_2 = 1$. Therefore the Green function is

$$g(t) = te^{-4t}.$$ 

(8) [9] Solve the initial-value problem

$$w'' + 8w' + 16w = \frac{8e^{-4t}}{1 + 4t^2}, \quad w(0) = w'(0) = 0.$$ 

Solution. By the last problem the Green function for this problem is $g(t) = te^{-4t}$. Because this equation is in normal form and because both the initial values are 0, the solution to this initial-value problem is given by the Green function formula

$$w(t) = \int_0^t g(t - s)f(s)\,ds = \int_0^t (t - s)e^{-4(t-s)} \frac{8e^{-4s}}{1 + 4s^2}\,ds$$

$$= te^{-4t} \int_0^t \frac{8}{1 + 4s^2}\,ds - e^{-4t} \int_0^t \frac{8s}{1 + 4s^2}\,ds.$$
Because
\[ \int_0^t \frac{8}{1 + 4s^2} \, ds = 4 \tan^{-1}(2s) \bigg|_{s=0}^t = 4 \tan^{-1}(2t), \]
\[ \int_0^t \frac{8s}{1 + 4s^2} \, ds = \log(1 + 4s^2) \bigg|_{s=0}^t = \log(1 + 4t^2), \]
we find that
\[ w(t) = 4t e^{-4t} \tan^{-1}(2t) - e^{-4t} \log(1 + 4t^2). \]

**Remark.** This problem can also be solved by using variation of parameters. However that approach is not as efficient because it does not directly solve the initial-value problem. Rather, after finding a particular solution the constants \( c_1 \) and \( c_2 \) in \( W_H(t) \) must be determined to satisfy the initial conditions.

(9) [10] The functions \( 1 - t \) and \( e^{-t} \) are solutions of the homogeneous equation
\[ tx'' - (1 - t)x' - x = 0 \quad \text{over } t > 0. \]

(You do not have to check that this is true!)

(a) [3] Show that these functions are linearly independent.

(b) [7] Give a general solution of the nonhomogeneous equation
\[ ty'' - (1 - t)y' - y = \frac{t^2}{1 - t} \quad \text{over } t > 0. \]

**Solution (a).** The Wronskian of \( 1 - t \) and \( e^{-t} \) is
\[ W[1 - t, e^{-t}](t) = \det \begin{pmatrix} 1 - t & e^{-t} \\ -1 & -e^{-t} \end{pmatrix} = (1 - t) \cdot (-e^{-t}) - (-1) \cdot e^{-t} = -e^{-t} + t e^{-t} + e^{-t} = t e^{-t}. \]

Because \( W[1 - t, e^{-t}](t) = te^{-t} \neq 0 \) for \( t > 0 \), the functions \( 1 - t \) and \( e^{-t} \) are linearly independent.

**Solution (b).** Because \( 1 - t \) and \( e^{-t} \) are linearly independent, a general solution of the associated homogeneous problem is
\[ y_H(t) = c_1 (1 - t) + c_2 e^{-t}. \]

Because this problem has variable coefficients, we should use either the general Green Function method or Variation of Parameters to find a particular solution \( y_P(t) \). Both of these methods require the equation to be put into its normal form, which is
\[ y'' - \frac{1 - t}{t} y' - \frac{1}{t} y = \frac{t}{1 - t}. \]

Notice that the forcing is not defined at \( t = 1 \).

**General Green Function.** The Green function \( G(t, s) \) is given by
\[ G(t, s) = \frac{1}{W[1 - s, e^{-s}]}(s) \det \begin{pmatrix} 1 - s & e^{-s} \\ 1 - t & e^{-t} \end{pmatrix} = \frac{(1 - s)e^{-t} - (1 - t)e^{-s}}{s e^{-s}}. \]
The Green function formula with any $t_I > 0$ such that $t_I \neq 1$ then yields the solution

$$y_p(t) = \int_{t_I}^{t} G(t, s) f(s) \, ds = \int_{t_I}^{t} \frac{(1-s)e^{-t} - (1-t)e^{-s}}{s e^{-s}} \frac{s}{1-s} \, ds$$

$$= e^{-t} \int_{t_I}^{t} e^s \, ds - (1-t) \int_{t_I}^{t} \frac{1}{1-s} \, ds .$$

We can evaluate the above definite integrals as

$$\int_{t_I}^{t} e^s \, ds = e^t \bigg|_{s=t_I}^{t} = e^t - e^{t_I},$$

$$- \int_{t_I}^{t} \frac{1}{1-s} \, ds = \log(|1-s|) \bigg|_{s=t_I}^{t} = \log(|1-t|) - \log(|1-t_I|) = \log\left|\frac{1-t}{1-t_I}\right| .$$

Therefore a general solution is

$$y(t) = c_1 (1-t) + c_2 e^{-t} + 1 - e^{t_I-t} + (1-t) \log\left|\frac{1-t}{1-t_I}\right| .$$

**Variation of Parameters.** We seek a solution in the form

$$y(t) = u_1(t)(1-t) + u_2(t)e^{-t} ,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$u_1'(t)(1-t) + u_2'(t)e^{-t} = 0 , \quad -u_1'(t) - u_2'(t)e^{-t} = \frac{t}{1-t} .$$

The solution of this system is

$$u_1'(t) = - \frac{1}{1-t} , \quad u_2'(t) = e^t .$$

Alternatively, because $W[1-t,e^{-t}](t) = t e^{-t}$, the formulas from the notes yield

$$u_1'(t) = \frac{e^{-t}}{t e^{-t}} \frac{t}{1-t} = - \frac{1}{1-t} , \quad u_2'(t) = \frac{1-t}{t e^{-t}} \frac{t}{1-t} = e^t .$$

No matter how they are obtained, you integrate these equations to find

$$u_1(t) = - \int \frac{1}{1-t} \, dt = c_1 + \log(|1-t|) ,$$

$$u_2(t) = \int e^t \, dt = c_2 + e^t .$$

Therefore a general solution is

$$y(t) = (c_1 + \log(|1-t|))(1-t) + (c_2 + e^t)e^{-t}$$

$$= c_1 (1-t) + c_2 e^{-t} + 1 + (1-t) \log(|1-t|) .$$

**Remark.** This general solution appears different than the one obtained by the general Green function method. However, replacing $c_1$ and $c_1$ in this solution with $c_1 - \log(|1-t_I|)$ and $c_2 - e^{t_I}$ transforms into the earlier one, so they are equivalent.
Find a particular solution \( v_P(t) \) of the equation \( v'' - 16v = 32e^{-4t} \).

**Solution.** This is a nonhomogeneous linear equation with constant coefficients. Its characteristic polynomial is

\[
p(z) = z^2 - 16 = (z + 4)(z - 4),
\]

which has two simple real roots \(-4\) and \(4\). The forcing \( 32e^{-4t} \) has characteristic form with degree \( d = 0 \) and characteristic \( \mu + i\nu = -4 \), which is a root of \( p(z) \) of multiplicity \( m = 1 \). Therefore we can use either Key Identity Evaluations or Undetermined Coefficients to find a particular solution \( v_P(t) \).

**Zero Degree Formula.** Because \( d = 0 \), \( \mu + i\nu = -4 \), and \( m = 1 \), we may use the zero degree formula with \( m = 1 \). Because \( p(z) = z^2 - 16 \), we see that \( p'(z) = 2z \), that \( p'(-4) = -8 \), and that

\[
L\left( \frac{t e^{-4t}}{p'(-4)} \right) = L\left( \frac{t e^{-4t}}{-8} \right) = e^{-4t}.
\]

Because \( L(v) = 32e^{-4t} \), we see that a particular solution is \( v_P(t) = -4te^{-4t} \).

**Remark.** Had you forgotten the zero degree formula then you could have derived it by Key Identity Evaluations as in the following solution.

**Key Indentity Evaluations.** Because \( m = m + d = 1 \) for the forcing \( 32e^{-4t} \), we only need the first derivative of the Key Identity. The Key Identity and its first derivative are

\[
L(e^{zt}) = (z^2 - 16)e^{zt},
\]
\[
L(te^{zt}) = (z^2 - 16)te^{zt} + 2ze^{zt}.
\]

By evaluating the first derivative of the Key identity at the characteristic \( z = -4 \) we obtain

\[
L(te^{-4t}) = 2 \cdot (-4) \cdot e^{-4t} = -8e^{-4t}.
\]

Because \( L(v) = 32e^{-4t} \), we see that a particular solution is \( v_P(t) = -4te^{-4t} \).

**Undetermined Coefficients.** Because \( \mu + i\nu = 3 \) and \( m = m + d = 1 \) for the forcing \( 32e^{-4t} \), we seek a particular solution in the form

\[
v_P(t) = At e^{-4t}.
\]

Because

\[
v'_p(t) = -4At e^{-4t} + Ae^{-4t}, \quad v''_p(t) = 16At e^{-4t} - 8Ae^{-4t},
\]

we obtain

\[
Lv_P(t) = [16At e^{-4t} - 8Ae^{-4t}] - 16[At e^{-4t}] = -8Ae^{-4t}.
\]

By setting \( Lv_P(t) = 32e^{-4t} \), we see that \(-8A = 32\), whereby \( A = -4 \). Therefore, a particular solution is \( v_P(t) = -4te^{-4t} \).

**Remark.** A general solution is \( v(t) = c_1e^{4t} + c_2e^{-4t} - 4te^{-4t} \).
The vertical displacement of an unforced mass on a spring is given by
\[ h(t) = -4e^{-5t} \cos(12t) - 3e^{-5t} \sin(12t). \]

(a) Is this system undamped, under damped, critically damped, or over damped? (Give your reasoning!)

(b) Express \( h(t) \) in the amplitude-phase form \( h(t) = Ae^{-5t} \cos(12t - \delta) \) with \( A > 0 \) and \( 0 \leq \delta < 2\pi \). Label the amplitude and phase. (The phase may be expressed in terms of an inverse trig function.)

(c) Give the natural frequency and natural period of this spring-mass system.

**Solution (a).** The system is *under damped* because the vertical displacement \( h(t) \) arises from a characteristic polynomial with the conjugate pair of roots \(-5 \pm i12\).

**Solution (b).** By comparing
\[ Ae^{-5t} \cos(12t - \delta) = Ae^{-5t} \cos(12t) + Ae^{-5t} \sin(\delta) \sin(12t), \]
with \( h(t) = -4e^{-5t} \cos(12t) - 3e^{-5t} \sin(12t) \), we see that
\[ A \cos(\delta) = -4, \quad A \sin(\delta) = -3. \]

This shows that \((A, \delta)\) are the polar coordinates of the point in the plane whose Cartesian coordinates are \((-4, -3)\). Clearly \(A\) is given by
\[ A = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5. \]
Because \((-4, -3)\) lies in the third quadrant, the phase \(\delta\) must satisfy \( \pi < \delta < \frac{3}{2} \pi \).

We can express \(\delta\) several ways. A picture shows that if we use \(\pi\) as a reference then
\[ \cos(\delta - \pi) = \frac{4}{5}, \quad \sin(\delta - \pi) = \frac{3}{5}, \quad \tan(\delta - \pi) = \frac{3}{4}, \]
whereby we can express the phase by any one of the formulas
\[ \delta = \pi + \cos^{-1}\left(\frac{4}{5}\right), \quad \delta = \pi + \sin^{-1}\left(\frac{3}{5}\right), \quad \delta = \pi + \tan^{-1}\left(\frac{3}{4}\right). \]
The same picture shows that if we use \(\frac{3}{2} \pi\) as a reference then
\[ \cos\left(\frac{3}{2} \pi - \delta\right) = \frac{3}{5}, \quad \sin\left(\frac{3}{2} \pi - \delta\right) = \frac{4}{5}, \quad \tan\left(\frac{3}{2} \pi - \delta\right) = \frac{4}{3}, \]
whereby we can express the phase by any one of the formulas
\[ \delta = \frac{3}{2} \pi - \cos^{-1}\left(\frac{3}{5}\right), \quad \delta = \frac{3}{2} \pi - \sin^{-1}\left(\frac{4}{5}\right), \quad \delta = \frac{3}{2} \pi - \tan^{-1}\left(\frac{4}{3}\right). \]

Only one expression for \(\delta\) is required.

**Remark.** It is incorrect to give the phase by one of the formulas
\[ \delta = \cos^{-1}\left( -\frac{4}{5}\right), \quad \delta = \sin^{-1}\left( -\frac{3}{5}\right), \quad \delta = \tan^{-1}\left(\frac{3}{4}\right), \]
because, by our conventions for the range of the inverse trigonometric functions, \(\cos^{-1}\left( -\frac{4}{5}\right)\) lies in \((\frac{\pi}{2}, \pi)\), \(\sin^{-1}\left( -\frac{3}{5}\right)\) lies in \((-\pi, 0)\), and \(\tan^{-1}\left(\frac{3}{4}\right)\) lies in \((0, \frac{\pi}{2})\).

**Solution (c).** Because the underlying characteristic polynomial has the conjugate pair of roots \(-5 \pm i12\), it must be
\[ p(z) = (z + 5)^2 + 12^2 = z^2 + 10z + 25 + 144 = z^2 + 10z + 169. \]
Therefore the vertical displacement \(h(t)\) satisfies the differential equation
\[ \ddot{h} + 10\dot{h} + 169h = 0. \]
We can read off that the natural frequency is \( \omega_0 = \sqrt{169} = 13 \) radians per sec, whereby the natural period \( T_0 \) is given by
\[
T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{13} \text{ sec}.
\]

(12) When a 10 gram mass is hung vertically from a spring, at rest it stretches the spring 5.0 cm. (Gravitational acceleration is \( g = 980 \text{ cm/sec}^2 \).) At \( t = 0 \) the mass is displaced 7 cm below its rest position and is released with a downward velocity of 3 cm/sec. The medium imparts a damping force of 900 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. Assume that the spring force is proportional to displacement, that the damping is proportional to velocity, and that there are no other forces.

(a) Formulate an initial-value problem that governs the motion of the mass for \( t > 0 \). (DO NOT solve this initial-value problem, just write it down!)

(b) Is this system undamped, under damped, critically damped, or over damped? (Give your reasoning!)

Solution (a). Let \( h(t) \) be the displacement (in centimeters) of the mass from its rest position at time \( t \) (in seconds), with upward displacements being positive. The governing initial-value problem then has the form
\[
m \ddot{h} + \gamma \dot{h} + kh = 0, \quad h(0) = -7, \quad \dot{h}(0) = -3,
\]
where \( m \) is the mass, \( \gamma \) is the damping coefficient, and \( k \) is the spring constant. We are given that \( m = 10 \) grams. We obtain \( k \) by balancing the force applied by the spring when it is stetched 5.0 cm with the weight of the mass (\( mg = 10 \cdot 980 \) dynes). This gives \( k \cdot 5.0 = 10 \cdot 980 \), or
\[
k = \frac{10 \cdot 980}{5.0} = 2 \cdot 980 \text{ dynes/cm}.
\]
We obtain \( \gamma \) by balancing the damping force when the speed of the mass is 4 cm/sec with 900 dynes. This gives \( \gamma \cdot 4 = 900 \), or
\[
\gamma = \frac{900}{4} \text{ dynes sec/cm}.
\]
Therefore the governing initial-value problem is
\[
10\ddot{h} + \frac{900}{4} \dot{h} + 2 \cdot 980h = 0, \quad h(0) = -7, \quad \dot{h}(0) = -3.
\]

Remark. Had we chosen the convention of downward displacements being positive then the governing initial-value problem is
\[
10\ddot{h} + \frac{900}{4} \dot{h} + 2 \cdot 980h = 0, \quad h(0) = 7, \quad \dot{h}(0) = 3.
\]

Solution (b). The normal form of the governing equation is
\[
\ddot{h} + \frac{90}{4} \dot{h} + 2 \cdot 98h.
\]
Its characteristic polynomial is
\[
p(z) = z^2 + \frac{90}{4}z + 196 = (z + \frac{45}{4})^2 + 196 - (\frac{45}{4})^2.
\]
Because \( 196 - (\frac{45}{4})^2 > 0 \), this polynomial has a conjugate pair of roots. Therefore the system is under damped.