(1) [5] Find the solution of the initial-value problem
\[
\frac{du}{dx} = -2xe^{-u}, \quad u(0) = u_I \quad \text{for some } u_I \text{ in } (-\infty, \infty).
\]
Give its interval of definition in terms of \( u_I \).

**Solution.** This differential equation is separable. Its separated differential form is
\[
e^u du = -2x \, dx,
\]
whereby
\[
\int e^u \, du = - \int 2x \, dx.
\]
Upon integrating both sides we find that
\[
e^u = -x^2 + c.
\]
The initial condition then implies that
\[
e^{u_I} = -0^2 + c, \quad \text{whereby } c = e^{u_I}.
\]
Therefore the solution satisfies
\[
e^u = e^{u_I} - x^2.
\]
This equation has a unique solution for \( u \) whenever \( e^{u_I} - x^2 > 0 \) that is given by
\[
u = \log(e^{u_I} - x^2).
\]
The condition \( e^{u_I} - x^2 > 0 \) is equivalent to \( e^{u_I} > x^2 \), which is then equivalent to
\[-e^{\frac{1}{2}u_I} < x < e^{\frac{1}{2}u_I}.
\]
Therefore the interval of definition is \((-\infty, e^{\frac{1}{2}u_I})\).

(2) [5] Sketch the phase-line portrait for the equation
\[
\frac{dz}{dt} = \frac{(z + 1)^2(z - 2)(z - 5)}{z^4 + 1}.
\]
Classify each stationary point as being either stable, unstable, or semistable. (You do not have to find the solution!)

**Solution.** The stationary points are found by setting
\[
\frac{(z + 1)^2(z - 2)(z - 5)}{z^4 + 1} = 0.
\]
Therefore the stationary points are \( z = -1, \ z = 2, \) and \( z = 5. \) Because \( z^4 + 1 > 0, \) a sign analysis of \((z + 1)^2(z - 2)(z - 5)\) shows that the phase-line portrait is
\[
\begin{array}{cccccc}
+ & + & - & + \\
\rightarrow & \rightarrow & \bullet & \rightarrow & \rightarrow & \bullet & \leftarrow & \leftarrow & \leftarrow & \bullet & \rightarrow & \rightarrow & \rightarrow & \rightarrow & z \\
-1 & 2 & 5 & \text{semistable} & \text{stable} & \text{unstable}
\end{array}
\]
**Remark.** If \( z(0) \) lies within the interval \([-1, 5]\) then the interval of definition of the solution \( z(t) \) is \((-\infty, \infty)\). Do you see why?