ENEE324, Home assignment 2. Date due September 24, 2025, 11:59pm EDT.

Instructor: Alexander Barg

Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. Suppose 2% of airline passengers carry a banned item, and a scanner catches 90% of those but also falsely flags 5% of passengers who follow the ban.

If you are flagged by the scanner, what is the probability you actually have a banned item?

Let B, L, C, F be the events of carrying a banned item, not carrying it being caught with a banned item, and falsely flagged.

Let A = flagged (falsely or not)

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{\overset{P(C)}{P(A|B)}\,P(B)}{P(A)} = \frac{P(C)\,P(B)}{P(A|B)P(B) + P(A|L)P(L)}$$

$$= \frac{0.9 \cdot 0.02}{0.9 \cdot 0.02 + 0.05 \cdot 0.98} = \frac{0.018}{0.018 + 0.049} = \frac{0.018}{0.229} \approx 0.268$$

Problem 2. From a standard 52-card deck, you draw 2 cards.

(a) What is the sample space of the experiment?

(b) You peek at the first and see it's a red card. What is the probability the second card is also red?

(c) What if instead someone tells you "at least one of your two cards is red"? What is the probability both are red now?

(a)-(b) The sample space is (all pairs of cards). Since one card is red, the number of choices of the form (red, red) equals # of remaining red cards $= 25$

$$P(R_2 | R_1) = \frac{25}{51} \approx 0.49$$

(b) The sample space is the same as before, but the number of favorable choices is different since we do not know which of the two cards is red.

$$P(\text{Both red} \mid \text{at least one is red}) = \frac{P(\text{Both red})}{P(\text{at least one red})}$$

$$\text{Numerator} = \frac{\binom{26}{2}}{\binom{52}{2}}$$

$$\text{Denominator} = \frac{|S| - |\overbrace{\{(B_1, B_2)\}}^{\text{Both Black}}|}{\binom{52}{2}} = 1 - \frac{\binom{26}{2}}{\binom{52}{2}}$$

$$\frac{\binom{26}{2} / \binom{52}{2}}{1 - \binom{26}{2} / \binom{52}{2}} \cong 0.324$$

Problem 3. A drawer contains 4 black socks, 4 white socks, and 2 red socks. You pick socks at random without looking.

(a) What is the probability the first two socks match?

(b) If the first sock is red, what is the probability the second sock matches it?

(c) Are events "first two socks match" and "first sock is red" independent?

(a) # of choices for the described event $= \binom{4}{2} + \binom{4}{2} + 1 = 13$

$$P(2 \text{ socks match}) = \frac{13}{\binom{10}{2}} = \frac{13}{45}$$

(b) Let R_1, R_2 be the events "1^{st} sock is red" and "2^{nd} sock is red".

$$P(R_2 | R_1) = \frac{1}{9 \text{ left}} = \frac{1}{9}$$

(c) Let M, R_1 be the two events in question

$P(M) = \frac{13}{45}$ from part (a) $P(R_1) = \frac{1}{5}$

$$P(M \cap R_1) = P(M | R_1) P(R_1) = \frac{1}{9} \cdot \frac{1}{5} = \frac{1}{45}$$

Compare with $P(M) P(R_1) = \frac{13}{225}$

Since $P(M, R_1) \neq P(M) P(R_1)$, these 2 events are not independent

Problem 4. (Variation of the Monty Hall problem) There are 4 doors; behind one door (chosen randomly) is a prize, the others are empty. You randomly pick one door. The host, who knows where the prize is, opens two empty doors, choosing them randomly among the remaining options. After that, the host offers you to switch from the original choice to the remaining closed door.

~~(a) What is the sample space of the described experiment?~~

(a) If you stick with your original choice, what is the probability of winning?

(b) If you switch to the remaining unopened door, what is the probability of winning?

(a)-(b) Assume without loss of generality that you chose D_1.

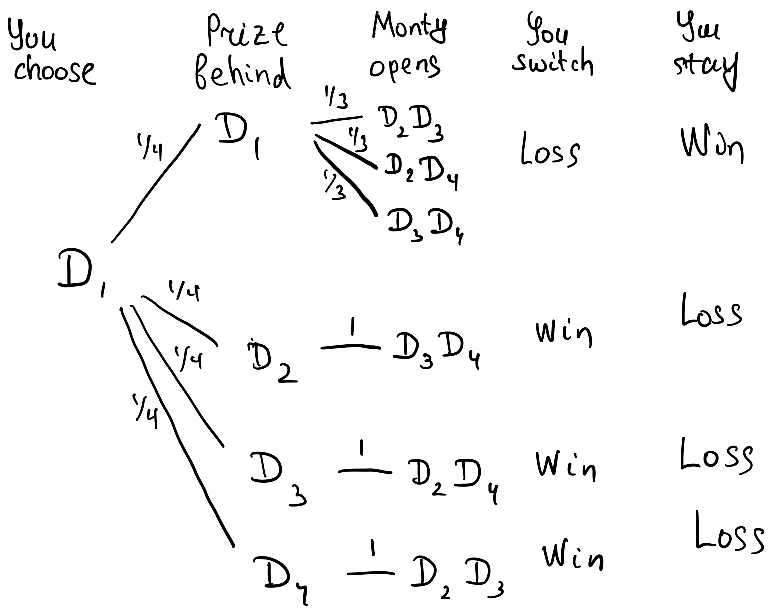
The sample space consists of $P \in \{1, 2, 3, 4\}$ (prize door) and

H = pair of doors opened by the host

P is uniform among $\{1,2,3,4\}$

If $P = 1$, H opens a uniformly chosen pair out of $\{2,3,4\}$

If $P \in \{2,3,4\}$, H opens the only pair of 2 empty doors



Thus if you switch, you win w. prob. $3/4$; without switching $P(\text{Win}) = \frac{1}{4}$

Problem 5. 100 passengers board a plane with 100 seats. The first passenger sits in a random seat. Each subsequent passenger takes their assigned seat if available; otherwise, they pick a random remaining seat.

(a) What is the probability the last passenger sits in their own seat? You cannot compute this directly (without quite some effort), so try to see what happens after the first passenger has chosen seat $j \in \{1, 2, \ldots, 100\}$.

(b) What is the probability that the second-to-last passenger sits in their own seat? First work out carefully the case of 4 passengers and 4 seats by case study, then generalize.

(c) Using any package, write a computer code to simulate this experiment. Run 1000 simulation cycles and compare the results with your calculations. Please include the code and the results in your submission.

(a) Without loss of generality we will assume that the correct seat assignment for passengers $P_1, P_2, \ldots, P_{100}$ is $\{1, 2, 3, \ldots, 100\}$.

Let j be the seat chosen by P_1; let $L = \{P_{100}$ in seat $100\}$

$j = 1 \Rightarrow L$ true

$j = 100 \Rightarrow L$ false

$1 < j < 100$: all passengers P_i, $i = 2, \ldots, j-1$ get their own seats. P_j chooses a random seat from $\{1, j+1, \ldots, 100\}$. This resets the process, albeit with fewer choices. Consider the last displaced passenger before P_{100}; since he's the last, he will have taken either 1 or 100 (or else there would be one more displaced before P_{100}). These choices are equally likely, so the answer is $1/2$.

(b) 4 passengers. $P(3^{rd} \text{ in } S_3) = \underbrace{\frac{1}{4} \cdot 1}_{P_1 \to S_1} + \underbrace{\frac{1}{4} \cdot \frac{2}{3}}_{P_1 - S_2;\ P_2 - S_1 \text{ or } S_4} + \underbrace{\frac{1}{4} \cdot 0}_{P_1 - S_3} + \underbrace{\frac{1}{4} \cdot 1}_{P_1 - S_4} = \frac{2}{3}$

Consider the moment when the first among seats 1, 3, 4 is taken. If this seat is 1 or 4, Pass. 3 gets own seat.

Follow the boarding until the first among seats 1, 99, 100 becomes occupied. By symmetry, each of (1, 99, 100) will be the first taken occurs with ~~prob~~ $1/3$. Call this seat T_1.

Suppose $T_1 = 1$ or N, then P_{99} gets own seat

If $T_1 = N-1$, P_{99} is out of luck.

Answer: $2/3$. See below for a computer simulation.

Problem 6. Alice is trying to send a message to Bob over a noisy communication channel. The message is encoded in binary, i.e., is a string of bits.

(a) Suppose A wants to send a single bit, 0 or 1, chosen with equal probabilities. 0 is transmitted correctly with probability 95% and flipped to 1 with prob. 5%. If A chooses to send 1, then there is a 10% chance of error, which results in B receiving 0. Assume that B received a 1, what is the probability that A actually sent a 1?

(b) Now suppose that A 'encodes' her message by sending 000 for the 0 message and 111 for the 1 message. The communication link flips (or not) each of the three transmitted bits independently as described in part (a) of this question. Given that B received 101, what is the probability that A intended to send a 1 as her message?

(a) Sample space : $(\text{bit}_A, \text{bit}_B)$

$$P(\text{bit}_A = 1 \mid \text{bit}_B = 1) = \frac{P(\text{bit}_A = \text{bit}_B = 1)}{P(\text{bit}_B = 1)}$$

$$= \frac{P(b_B = 1 \mid b_A = 1)\, \cancel{P(b_A = 1)}}{P(b_B = 1 \mid b_A = 1)\, \cancel{P(b_A = 1)} + P(b_b = 1 \mid b_A = 0)\, \cancel{P(b_A = 0)}}$$

$$= \frac{0.9}{0.9 + 0.05} = \frac{0.9}{0.95} \cong 0.95$$

(b) $P(B = 101 \mid b_A = 1) = P(B = 101 \mid A = 111) = 0.9 \cdot 0.1 \cdot 0.9 = 0.081$

$$P(b_A = 1 \mid B = 101) = \frac{P(B = 101 \mid b_A = 1) \cdot \cancel{P(b_A = 1)}}{P(B = 101 \mid b_A = 1) \cdot \cancel{P(b_A = 1)} + P(B = 101 \mid b_A = 0)\, \cancel{P(b_A = 0)}}$$

$$= \frac{0.081}{0.081 + 0.05^2 \cdot 0.95} \approx 0.97.$$

Simulation of the experiment in the airplane seating problem, Prob. 5(b).
With 100K independent trials, the 2/3 answer is closely approximated

```
n = 100; numTrials = 10^5;

SeedRandom[1234];

simulateOnce[] := Module[{availableSeats, seating, firstChoice, i, chosenSeat},
    availableSeats = Range[n];
    seating = ConstantArray[None, n];
    firstChoice = RandomChoice[availableSeats];
    seating[[1]] = firstChoice;
    availableSeats = DeleteCases[availableSeats, firstChoice];
    (*Remaining passengers 2 to n*)For[i = 2, i <= n, i++,
      If[MemberQ[availableSeats, i], chosenSeat = i, chosenSeat = RandomChoice[availableSeats]];
      seating[[i]] = chosenSeat;
      availableSeats = DeleteCases[availableSeats, chosenSeat];];
      seating[[n - 1]] === n - 1]

(*Monte Carlo estimate*)
probEstimate = N[Mean@Table[If[simulateOnce[], 1, 0], {numTrials}]]

0.66701
```