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- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points.

Problem 1. A semiconductor fab finds that 2% of manufactured chips are defective. An automated tester flags a defective chip as "bad" with probability 0.98, and flags a good chip as "bad" with probability 0.05.

(a) A randomly selected chip is flagged as "bad". What is the probability that it is actually defective?

(b) Suppose the chip is tested twice, with the two tests conditionally independent given the true state of the chip, and both tests flag it as "bad". What is now the probability that the chip is defective?

Solution. Let D be the event "defective" and B the event "flagged bad". We are given $P(D) = 0.02$, $P(B | D) = 0.98$, $P(B | D^c) = 0.05$.

(a) By the law of total probability,

$$P(B) = P(B | D)P(D) + P(B | D^c)P(D^c) = 0.98(0.02) + 0.05(0.98) = 0.0196 + 0.049 = 0.0686.$$

By Bayes' rule,

$$P(D | B) = \frac{P(B | D)P(D)}{P(B)} = \frac{0.0196}{0.0686} \approx 0.2857.$$

(b) With two conditionally independent tests, $P(BB | D) = 0.98^2 = 0.9604$ and $P(BB | D^c) = 0.05^2 = 0.0025$. Then

$$P(D | BB) = \frac{0.9604(0.02)}{0.9604(0.02) + 0.0025(0.98)} = \frac{0.019208}{0.019208 + 0.00245} \approx 0.8869.$$

Two agreeing positives raise the posterior from about 29% to about 89%.

Problem 2. A firm sources one electronic component from three suppliers S_1, S_2, S_3 , which provide 50%, 30%, and 20% of the components, respectively. The fraction of defective components is 1% from S_1 , 2% from S_2 , and 4% from S_3 .

(a) What is the probability that a randomly chosen component is defective?

(b) Given that a component is defective, what is the probability that it came from supplier S_1 ?

(c) Let E_1 be the event "the component came from S_1 " and E_2 the event "the component is defective". Is it true that $P(E_1)P(E_2) = P(E_1E_2)$?

(d) Two components are chosen independently at random. What is the probability that they come from different suppliers?

Solution. Write $P(S_1) = 0.5$, $P(S_2) = 0.3$, $P(S_3) = 0.2$ and $P(D | S_1) = 0.01$, $P(D | S_2) = 0.02$, $P(D | S_3) = 0.04$.

(a) By total probability,

$$P(D) = 0.5(0.01) + 0.3(0.02) + 0.2(0.04) = 0.005 + 0.006 + 0.008 = 0.019.$$

(b) By Bayes' rule,

$$P(S_1 | D) = \frac{P(D | S_1)P(S_1)}{P(D)} = \frac{0.005}{0.019} \approx 0.2632.$$

(c) $P(E_1 E_2) = P(D | S_1)P(S_1) = 0.005$, while $P(E_1)P(E_2) = 0.5(0.019) = 0.0095$. Since $0.005 \neq 0.0095$, the events are *not* independent.

(d) $P(\text{different suppliers}) = 1 - (P(S_1)^2 + P(S_2)^2 + P(S_3)^2) = 1 - (0.25 + 0.09 + 0.04) = 1 - 0.38 = 0.62$.

Problem 3. A family has two children. Assume each child is a boy or a girl with probability $1/2$, independently of the other, and that boys and girls are equally likely.

(a) Given that at least one of the children is a girl, what is the probability that both children are girls?

(b) Given that the elder child is a girl, what is the probability that both children are girls?

(c) The two answers differ even though “at least one is a girl” and “the elder is a girl” may sound similar. Explain carefully why conditioning on these two events gives different probabilities.

Solution. The sample space of ordered pairs (elder, younger) is $\{BB, BG, GB, GG\}$, each with probability $1/4$.

(a) The event “at least one girl” is $\{BG, GB, GG\}$ (probability $3/4$); “both girls” is $\{GG\}$. Hence

$$P(GG | \text{at least one girl}) = \frac{1/4}{3/4} = \frac{1}{3}.$$

(b) The event “elder is a girl” is $\{GB, GG\}$ (probability $1/2$). Hence

$$P(GG | \text{elder is a girl}) = \frac{1/4}{1/2} = \frac{1}{2}.$$

(c) Conditioning restricts the sample space to the outcomes consistent with the information. “At least one girl” leaves three equally likely outcomes $\{BG, GB, GG\}$, only one of which is GG , giving $1/3$. “The elder is a girl” is more specific: it fixes one designated child and leaves only $\{GB, GG\}$, so the remaining (younger) child is a girl with probability $1/2$. Identifying a particular child removes the outcome BG from consideration, which changes the conditional probability.

Problem 4. Two evenly matched players, the server S and the receiver R , are at *deuce* in a tennis game. On each point the server wins with probability p and the receiver wins with probability $1 - p$, independently of all other points. Starting from deuce, a player wins the game as soon as they lead by two points; if the two players ever return to an equal score, that state is again a deuce.

- (1) Let $w = P(\text{server wins the game} | \text{currently at deuce})$. Condition on the outcome of the *next two points* (the point played by each side). List the three possible outcomes of these two points and their probabilities.
- (2) Explain why, if the next two points are split (one won by each player), the situation is again exactly a deuce, so the server still wins with probability w .
- (3) Combine the previous steps into an equation for w of the form

$$w = (\text{probability } S \text{ wins the next two points}) + (\text{probability the next two points are split}) \cdot w.$$

- (4) Solve for w and evaluate it for $p = 0.6$. Does the server have an advantage at deuce?
- (5) Simulate the game from deuce by computer to confirm your answer for $p = 0.6$. What is the smallest number of trials you needed to match the exact answer to within 10^{-2} ? To within 10^{-3} ? You may use any software package; include your code in your write-up.

Solution. (1) Consider the next two points. Exactly three things can happen: S wins both, with probability p^2 ; R wins both, with probability $(1-p)^2$; the points are split (one each), with probability $2p(1-p)$.

(2) If the players split the next two points, the score returns to equal, i.e. to deuce, and by the memorylessness of independent points the game ahead is statistically identical to the game that started at the original deuce. Hence the server's winning probability from that point is again w .

(3) If S wins both of the next two points the game is won outright; if the points are split the server wins with probability w ; if R wins both the server loses. Therefore

$$w = p^2 + 2p(1-p)w.$$

(4) Solving,

$$w(1 - 2p(1-p)) = p^2 \implies w = \frac{p^2}{1 - 2p(1-p)} = \frac{p^2}{p^2 + (1-p)^2}.$$

For $p = 0.6$, $w = \frac{0.36}{0.36 + 0.16} = \frac{0.36}{0.52} = \frac{9}{13} \approx 0.6923$. So at deuce a point-advantage of 0.6 is amplified: the server wins the game with probability about 0.69.

(5) A Monte Carlo simulation of the deuce game with $p = 0.6$ produces a fraction of server wins that converges to $9/13 \approx 0.6923$. Because the estimate is an average of independent Bernoulli trials, its typical error decreases like $N^{-1/2}$: matching the exact value to within 10^{-2} requires on the order of 10^3 – 10^4 trials, while matching to within 10^{-3} requires roughly two orders of magnitude more (on the order of 10^5 – 10^6 trials). Exact counts depend on the random seed.

Problem 5. An urn contains 5 green marbles and 3 yellow marbles. Four marbles are drawn.

(a) If the marbles are drawn without replacement, what is the probability that exactly two of the four are green?

(b) Still drawing without replacement, given that at least one of the four marbles is yellow, what is the probability that exactly one of the four is yellow?

(c) Suppose instead the marbles are drawn *with* replacement (each draw independent, green with probability $5/8$). Recompute the probabilities in (a) and (b) under this model.

Solution. There are $\binom{8}{4} = 70$ equally likely subsets when drawing without replacement.

(a) “Exactly two green” means two green and two yellow:

$$\frac{\binom{5}{2}\binom{3}{2}}{\binom{8}{4}} = \frac{10 \cdot 3}{70} = \frac{30}{70} = \frac{3}{7} \approx 0.4286.$$

(b) “All four green” has probability $\binom{5}{4}/\binom{8}{4} = 5/70$, so $P(\text{at least one yellow}) = 1 - \frac{5}{70} = \frac{65}{70}$. “Exactly one yellow” is $\binom{3}{1}\binom{5}{3}/\binom{8}{4} = \frac{3 \cdot 10}{70} = \frac{30}{70}$. Therefore

$$P(\text{exactly one yellow} \mid \text{at least one yellow}) = \frac{30/70}{65/70} = \frac{30}{65} = \frac{6}{13} \approx 0.4615.$$

(c) Now each draw is green with probability $5/8$ and yellow with probability $3/8$, independently, so the number of yellows in 4 draws is Binomial(4, $3/8$).

Exactly two green (two yellow): $\binom{4}{2}\left(\frac{5}{8}\right)^2\left(\frac{3}{8}\right)^2 = 6 \cdot \frac{25}{64} \cdot \frac{9}{64} = \frac{1350}{4096} = \frac{675}{2048} \approx 0.3296$.

For (b): $P(\text{all green}) = \left(\frac{5}{8}\right)^4 = \frac{625}{4096}$, so $P(\text{at least one yellow}) = \frac{3471}{4096}$. Exactly one yellow = $\binom{4}{1}\left(\frac{3}{8}\right)\left(\frac{5}{8}\right)^3 = \frac{1500}{4096}$. Hence

$$P(\text{exactly one yellow} \mid \text{at least one yellow}) = \frac{1500/4096}{3471/4096} = \frac{1500}{3471} = \frac{500}{1157} \approx 0.4322.$$

Problem 6. A message bit $X \in \{0, 1\}$ is transmitted over a noisy channel. The sender chooses $X = 1$ with probability 0.4 and $X = 0$ with probability 0.6. The channel flips a transmitted bit independently with probability 0.1.

(a) The receiver observes $Y = 1$. Find the posterior probability $P(X = 1 \mid Y = 1)$.

(b) Suppose each bit is transmitted three times independently (i.e., X is sent as XXX) and the receiver decodes by majority rule. What is the probability that the receiver decodes the bit incorrectly?

(c) Compare the error probability in part (b) with the error probability of a single transmission.

Solution. (a) The channel gives $P(Y = 1 \mid X = 1) = 0.9$ and $P(Y = 1 \mid X = 0) = 0.1$. By total probability,

$$P(Y = 1) = 0.9(0.4) + 0.1(0.6) = 0.36 + 0.06 = 0.42,$$

and by Bayes' rule

$$P(X = 1 \mid Y = 1) = \frac{0.9(0.4)}{0.42} = \frac{0.36}{0.42} = \frac{6}{7} \approx 0.857.$$

(b) With three independent transmissions, majority decoding fails exactly when at least two of the three bits are flipped. Since flips are independent with probability 0.1,

$$P(\text{error}) = \binom{3}{2}(0.1)^2(0.9) + (0.1)^3 = 3(0.009) + 0.001 = 0.027 + 0.001 = 0.028.$$

(This value does not depend on the transmitted symbol, because the channel is symmetric.)

(c) A single transmission is decoded incorrectly with probability 0.1. Repetition with majority decoding reduces the error probability from 0.1 to 0.028, roughly a 3.6-fold improvement.