ENEE324, Home assignment 3. Date due September 24, 2025, 11:59pm EDT.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.

- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. Suppose that there are 5 blood types in the population, named type 1 through type 5, with probabilities $p_1, p_2, \ldots, p_5$. A crime was committed by two individuals. A suspect, who has blood type 1, has prior probability p of being guilty. At the crime scene, blood evidence is collected, which shows that one of the criminals has type 1 and the other has type 2.

Find the posterior probability that the suspect is guilty, given the evidence. Does the evidence make it more likely or less likely that the suspect is guilty, or does this depend on the values of the parameters $p, p_1, p_2, \ldots, p_5$? If it depends on these values, give a simple criterion for when the evidence makes it more likely that the suspect is guilty.

Hint: Use the Bayes formula for the conditional probability in question.

1. Denote by G the event "suspect guilty"; G^c means "innocent." Let E refer to the evidence, blood types 1 and 2 collected at the scene.

$$P(G|E) = \frac{P(E|G)\,P(G)}{P(E)} = \frac{P(E|G)\,\overbrace{P(G)}^{=p}}{P(E|G)\,P(G) + P(E|G^c)\,P(G^c)}$$

Let us analyze $P(E|G)$. Conditional on G, type 1 is assured, so $P(E|G)$ = probability of finding type 2 at the scene

$= p_2$

$P(E|G^c)$ = prob. that the 2 perpetrators have types 1 and 2

$= p_1 p_2 + p_2 p_1 = 2p_1 p_2$

Altogether $$P(G|E) = \frac{p_2\, p}{p_2\, p + 2p_1 p_2 (1-p)}$$

We are asked to resolve the question $P(G|E) \gtrless P(G) = p$

$$\frac{P(G|E)}{P(G)} = \frac{p_2}{p_2 p + 2p_1 p_2(1-p)} = \frac{1}{p + 2p_1(1-p)}$$

Assume that this is >1 (i.e., the evidence adds plausibility to "guilty"), i.e. $p + 2p_1(1-p) < 1$ or $p_1 < 1/2$.

Thus, if $p_1 < \frac{1}{2}$, the prob. of "guilty" rises based on the evidence. Indeed, if type 1 is rare, it is likely that the suspect was involved.

Problem 2. You're at a party with 199 other guests when robbers break in and announce that they are going to rob one of you. They put 199 blank pieces of paper in a hat, plus one marked "you lose." Each guest must draw, and the person who draws "you lose" will get robbed. The robbers offer you the option of drawing first, last, or at any time in between. When would you take your turn?

The draws are made without replacement, and for (a) are uniformly random.

(a) Determine whether it is optimal to draw first, last, or somewhere in between (or whether it does not matter), to maximize the probability of not being robbed. To answer this, compute P_k, the probability of the loss if drawing kth, and then examine how these numbers change with k.

(b) More generally, suppose that there is one "you lose" piece of paper, with "weight" v, and there are $n \geq 199$ blank pieces of paper, each with "weight" w. At each stage, draws are made with probability proportional to weight, i.e., the probability of drawing a particular piece of paper is its weight divided by the sum of the weights of all the remaining pieces of paper. Determine whether it is better to draw first or second (or whether it does not matter); here $v > 0$; $w > 0$; and $n \geq 1$ are known constants. Use the same approach as in Part (a) and note that your answer may depend on the relation between v and w.

2 (a) Ans.: it does not matter. Let $m = 199$ be # of guests.

$$P_1 = \text{prob. of loss if drawing } 1^{st} = \frac{1}{m+1}$$

Now note that if you go 2^{nd} (or k^{th}), then the first one to draw (the first $k-1$ to draw) picked a blank slip.

$$P_2 = P(\text{loss}, \text{blank}_1) = P(\text{loss} \mid \text{blank}_1) P(\text{blank}_1)$$

$$= \frac{1}{m} \cdot \frac{m}{m+1} = \frac{1}{m+1}$$

Likewise

$$P_k = P(\text{loss}, \text{Blank}_1, \dots, \text{Blank}_{k-1}) = \frac{1}{(m+1)-k} \cdot \underbrace{\frac{(m+1)-(k-2)}{(m+1)-(k-1)} \cdots \frac{m}{m+1}}_{\prod_{i=0}^{k-1} \frac{(m+1)-(i-1)}{(m+1)-i}} = \frac{1}{m+1}$$

for all k.

How is it possible that, even if you draw last, and there are only two slips left, your prob. of loss is still $\frac{1}{m+1}$?
The reason is that the probability that none of the guests before you drew the "loss" slip is that low.

(b) Let $z = \frac{v}{w}$. We have, as above,

$$P_1 = \frac{v}{v+nw} = \frac{z}{n+z}$$

$$P_2 = \underbrace{\frac{v}{v+(n-1)w}}_{\text{your loss}} \; \underbrace{\frac{wn}{v+nw}}_{1^{st}\text{ draw Blank}} = \frac{z}{(n-1)+z} \cdot \frac{n}{n+z}$$

Consider

$$\frac{P_2}{P_1} = \frac{z}{n-1+z} \, \frac{n}{n+z} \, \frac{n+z}{z} = \frac{n}{n+z-1} = \frac{1}{1+\frac{z-1}{n}}$$

If $z<1$, then $P_1 < P_2$, so going 1^{st} is better;
if $z>1$, it is the opposite
If $z=1$, we are back to part (a), so it does not matter

Problem 3. There are two identical coins with $P(H) = p, P(T) = 1 - p$. Player 1 repeatedly tosses the first coin, and simultaneously Player 2 repeatedly tosses the second coin. Define two independent random variables, X_1 and X_2, where $X_i, i = 1, 2$ is the number of tosses until Player i tosses H *for the first time*, not counting H itself. Consider new random variables, $Y = \max(X_1, X_2)$, $Z = \min(X_1, X_2)$. Find the pmf's of Y and Z.

3. (a) $P(X_1 \geq n) = P(\text{first } n \text{ tosses did not yield an } H) = (1-p)^n$

or $= \sum_{k=n}^{\infty} (1-p)^k p = p \frac{(1-p)^n}{p} = (1-p)^n$

For the last equality we used $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ geometric summation

(b) Start with $Y = \max(X_1, X_2)$. We need to find

$$p_Y(n) = P(Y=n) = P(Y \leq n) - P(Y \leq n-1)$$

$$P(Y \leq n) = (P(X_1 \leq n))^2 = \left(\sum_{k=0}^{n} (1-p)^k p\right)^2 = \left(1-(1-p)^{n+1}\right)^2$$

$$P(Y \leq n-1) = \left(1-(1-p)^n\right)^2$$

$$P(Y=n) = \left(1-(1-p)^{n+1}\right)^2 - \left(1-(1-p)^n\right)^2, \quad n \geq 0$$

Now consider $Z = \min(X_1, X_2)$

$$p_Z(n) = P(Z \geq n) - P(Z \geq n+1)$$

$$P(Z \geq n) = P(X_1 \geq n \text{ and } X_2 \geq n) \overset{\text{independence}}{=} \left(p \sum_{k=n}^{\infty} (1-p)^k\right)^2$$

$$= (1-p)^{2n} \leftarrow \text{inf. geometric sum } \sum_{k \geq n} x^k = \frac{x^n}{1-x} \text{ if } |x| < 1$$

$$p_Z(n) = (1-p)^{2n} - (1-p)^{2n+2} = (1-p)^{2n}\left(1-(1-p)^2\right) = (1-p)^{2n} p(2-p)$$

$$= \left((1-p)^2\right)^n p(2-p) = \left(1-(p(2-p))\right)^n p(2-p), \quad n \geq 0$$

Note that $Z \sim \text{geom}(p(2-p))$

Problem 4. 4 people, each wearing a hat, come to a party. When they leave, their 4 hats are assigned to them randomly, meaning that each permutation is equally likely. Let X be the random number of people who got their own hat. Find the pmf of X.

This can be done by explicitly examining all the possibilities; however you are expected to present an argument that supports the derivation of the results. Certainly, for a general number, n, of guests, this is the only way to do this.

4. (a) Following the hint in the assignment, fix a subset of m guests, none of whom get their own hat. Let T_k = assignments that give the k^{th} guest his own hat. By inclusion-exclusion,

$$|T_1 \cup T_2 \cup \dots \cup T_m| = \sum_i |T_i| - \sum_{i<j} |T_i \cap T_j| + \sum_{i<j<k} |T_i \cap T_j \cap T_k|$$
$$- \dots + (-1)^{m+1} |T_1 \cap T_2 \cap \dots \cap T_m|$$
$$= \binom{m}{1}(m-1)! - \binom{m}{2}(m-2)! + \binom{m}{3}(m-3)! - \dots + (-1)^{m+1}\binom{m}{m}0!$$

\# terms in the sum

$$= \sum_{k=1}^{m} (-1)^{m+1} \binom{m}{k}(m-k)!$$

The number of assignments D_m in which none get their own hat is all the remaining asignments:

$$D_m = m! - \sum_{k=1}^{m} (-1)^{m+1}\binom{m}{k}(m-k)! = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!}$$

(b) Next, we can choose k guests that get their own hat in $\binom{n}{k}$ ways

So # assignments with exactly k guests getting their hat is $\binom{n}{k} D_{n-k}$.

$$P(k \text{ get their hats}) = \frac{\binom{n}{k} D_{n-k}}{n!}, \quad k = 0, 1, \dots, n$$

with $n=4$, we obtain

k	0	1	2	3	4
$p(k)$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{1}{4}$	0	$\frac{1}{24}$

why 0? Because if 3 get their own hats, then so does the 4th

Problem 5. A factory produces light bulbs, each defective with probability $p = 0.05$, independently. A quality inspector draws $n = 10$ bulbs at random from a very large batch.

- If he finds **two or more defectives**, ~~s~~he **rejects the batch** immediately.
- If he finds **no defectives**, ~~s~~he **accepts the batch** immediately.
- If he finds **exactly one defective**, ~~s~~he draws another $m = 5$ bulbs and **accepts the batch if all of them are good**, otherwise rejects.

Let $X \sim \text{Binomial}(n, p)$ be the number of defectives in the first sample, and $Y \sim \text{Binomial}(m, p)$ be the number of defectives in the second sample (if needed).

(1) Find the pmf X.
(2) Compute $P(\text{batch accepted})$.
(3) Compute $P(\text{batch rejected})$ and check that the probabilities sum to 1.

(1) $X \sim \text{Binomial}(n, p)$

$$p_X(k) = \binom{10}{k} 0.05^k \, 0.95^{10-k}, \quad k = 0, 1, \dots, 10$$

(2) $$P(\text{accepted}) = P(X=0) + P(Y=0, X=1) \overset{\text{independence}}{=} P(X=0) + P(Y=0)\, P(X=1)$$

$$= p_X(0) + p_Y(0)\, p_X(1) = 0.95^{10} + 0.95^5 \cdot (10 \times 0.05 \times 0.95^9)$$

$$\approx 0.842$$

(3) $$P(\text{rejected}) = P(X \geqslant 2) + P(Y \geqslant 1)\, P(X=1)$$

$$= (1 - P(X \leq 1)) + (1 - P(Y=0))\, P(X=1)$$

$$= (1 - 0.95^{10} - 10 \times 0.05 \times 0.95^9) + (1 - 0.95^5)(10 \times 0.05 \times 0.95^9)$$

$$\approx 0.157$$

The numbers obtained in (2) and (3) add to 1 (up to the rounding error)

See the problem statement in the assignment

~~**Problem 6.** A sensor reports a binary signal ("0" or "1"). Each time you read it, the reading is independently wrong with probability $p = 0.1$. To improve reliability, you read the sensor $n = 5$ times and take the **majority vote**.~~

~~(1) Let $X \sim \text{Binomial}(n, p)$ be the number of erroneous readings. Write $P(\text{majority vote is wrong})$ in terms of the pmf of X.~~
~~(2) Compute the numerical value of $P(\text{majority vote is wrong})$ for $p = 0.1$.~~
~~(3) What is the smallest odd n such that $P(\text{majority vote is wrong}) < 0.01$?~~

(1) pmf of X $\quad p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{5}{k} 0.1^k \, 0.9^{5-k}, \quad k = 0, 1, \dots, 5$

$$P(\text{majority is wrong}) = \sum_{k > \frac{n}{2}}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

sum from $\lceil \frac{n}{2} \rceil$ if n odd
$\frac{n}{2} + 1$ if n is even

To account for more incorrect than correct readings, we sum from $k = \lceil \frac{n}{2} \rceil$ if n is odd and $k = \frac{n}{2} + 1$ if n is even; in both cases the largest $k = n$

(2) $$\sum_{k \geq 3}^{5} \binom{5}{k} 0.1^k (1-p)^{5-k} \approx 0.0086$$

(3) $$P(\text{majority is wrong}) = \sum_{k = \lceil \frac{n}{2} \rceil} \binom{n}{k} p^k (1-p)^{n-k}$$

Take $p = 0.2$, then the first odd n for which this sum < 0.01 is $n = 13$. The value $P(\text{majority is wrong})_{n=13} \approx 0.007$