ENEE324, Home assignment 3. Date due October 5, 2025, 11:59pm EDT.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. Suppose that there are 5 blood types in the population, named type 1 through type 5, with probabilities $p_1, p_2, \ldots, p_5$. A crime was committed by two individuals. A suspect, who has blood type 1, has prior probability p of being guilty. At the crime scene, blood evidence is collected, which shows that one of the criminals has type 1 and the other has type 2.

Find the posterior probability that the suspect is guilty, given the evidence. Does the evidence make it more likely or less likely that the suspect is guilty, or does this depend on the values of the parameters $p, p_1, p_2, \ldots, p_5$? If it depends on these values, give a simple criterion for when the evidence makes it more likely that the suspect is guilty.

Hint: Use the Bayes formula for the conditional probability in question.

Problem 2. You're at a party with 199 other guests when robbers break in and announce that they are going to rob one of you. They put 199 blank pieces of paper in a hat, plus one marked "you lose." Each guest must draw, and the person who draws "you lose" will get robbed. The robbers offer you the option of drawing first, last, or at any time in between. When would you take your turn?

The draws are made without replacement, and for (a) are uniformly random.

(a) Determine whether it is optimal to draw first, last, or somewhere in between (or whether it does not matter), to maximize the probability of not being robbed. To answer this, compute P_k, the probability of the loss if drawing kth, and then examine how these numbers change with k.

(b) More generally, suppose that there is one "you lose" piece of paper, with "weight" v, and there are n blank pieces of paper, each with "weight" w. At each stage, draws are made with probability proportional to weight, i.e., the probability of drawing a particular piece of paper is its weight divided by the sum of the weights of all the remaining pieces of paper. Determine whether it is better to draw first or second (or whether it does not matter); here $v > 0; w > 0$; and $n \geq 1$ are known constants. Use the same approach as in Part (a) and note that your answer may depend on the relation between v and w.

Problem 3. There are two identical coins with $P(H) = p, P(T) = 1 - p$. Player 1 repeatedly tosses the first coin, and simultaneously Player 2 repeatedly tosses the second coin. Define two independent random variables, X_1 and X_2, where $X_i, i = 1, 2$ is the number of tosses until Player i tosses H *for the first time*, not counting H itself. Such RVs are called *geometric random variables* with parameter p.

(a) What is $P(X_1 \geq n)$ for some fixed number n?

(b) Find the probability mass functions of the random variables, $Y = \max(X_1, X_2)$, $Z = \min(X_1, X_2)$. Hint: If $Y \leq n$ then both $X_1, X_2 \leq n$. If $Z \geq n$, then both $X_1, X_2 \geq n$.

Problem 4. 4 people, each wearing a hat, come to a party. When they leave, their 4 hats are assigned to them randomly, meaning that each permutation is equally likely. Let X be the random number of people who got their own hat. Find the pmf of X.

Since there are only 4!=24 possible assignments, this can be done by explicitly examining all the possibilities by computer, but doing this will earn you no credit (although you can check your answers in that way). You are expected to present an argument that supports the derivation of the results for an arbitrary n.

Hint: Say guests 1,2,3,4 are assigned hats 1,3,4,2 (in this order). Then 1 guest gets their own hat. There are $\binom{4}{1}$ ways to choose the guest that gets their own hat. It remains to find the number of permutations of m elements in which none get their own hat; in the example above, $m = 3$. This number equals the total number of assignments, $m!$, minus the number of assignments in which at least one gets their own hat. This last quantity is found by a direct application of inclusion-exclusion. Having done this, we obtain the count of assignments with k matches (above, $k = 1$), and the probability of k matches is this count divided by $4!$.

Problem 5. A factory produces light bulbs, each defective with probability $p = 0.05$, independently. A quality inspector draws $n = 10$ bulbs at random from a very large batch.

- If he finds two or more defectives, he rejects the batch immediately.
- If he finds no defectives, he accepts the batch immediately.
- If he finds exactly one defective, he draws another $m = 5$ bulbs and accepts the batch if all of them are good, otherwise rejects.

Let $X \sim \text{Binomial}(n, p)$ be the number of defectives in the first sample, and $Y \sim \text{Binomial}(m, p)$ be the number of defectives in the second sample (if needed).

(1) Find the pmf X.
(2) Compute $P(\text{batch accepted})$.
(3) Compute $P(\text{batch rejected})$ and check that the probabilities sum to 1.

Problem 6. A sensor reports a binary signal ("0" or "1"). Each time you read it, the reading is independently wrong with probability p. To improve reliability, you read the sensor n times and take the majority vote.

(1) Let $X \sim \text{Binomial}(n, p)$ be the number of erroneous readings. Write $P(\text{majority vote is wrong})$, i.e., that there are more erroneous than correct readings, in terms of the pmf of X for general n and p.
(2) Compute the numerical value of $P(\text{majority vote is wrong})$ for $n = 5, p = 0.1$.
(3) Take $p = 0.2$. What is the smallest odd n such that $P(\text{majority vote is wrong}) < 0.01$?