ENEE324, Home assignment 3. Date due March 2, 2026, 11:59pm EDT.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.

- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. In a certain city there are four taxi companies, labeled A, B, C, D, which operate 40%, 30%, 20%, and 10% of all taxis, respectively. Company A uses electric cars 50% of the time, while the corresponding proportions for B, C, D are 30%, 20%, and 10%.

A hit-and-run accident was committed by an electric taxi. A witness identifies the taxi as belonging to company A. The witness is correct in identifying the company with probability 0.8 (and when incorrect, each of the other three companies is chosen with equal probability).

(1) Find the posterior probability that the taxi was from company A.
(2) Does the witness testimony increase or decrease the probability that the taxi was from A compared to the prior probability (the probability that the taxi was from A without relying on the information from the witness)? Explain.
(3) Give a condition (in terms of the electric-car proportions and fleet sizes) under which such testimony would *decrease* the probability that the taxi was from A.

Let E = electric car, W = witness correct

$$\underline{P(E)} = P(E|A)P(A) + P(E|B)P(B) + P(E|C)P(C) + P(E|D)P(D)$$

$$= 0.5\cdot 0.4 + 0.3\cdot 0.3 + 0.2\cdot 0.2 + 0.1\cdot 0.1 = 0.2 + 0.09 + 0.04 + 0.01 = 0.34$$

Pt. (1) $$P(A|E,W) = \frac{P(A,E,W)}{P(WE)} = \frac{P(W|A,E)\,P(A,E)}{P(WE)}$$

$$= \overset{1}{P(W|A,E)}\cdot \overset{2}{P(AE)}\,\overset{3}{\frac{1}{P(WE)}}$$

$$= \overset{1}{P(W|A)}\;\overset{2}{P(E|A)P(A)} \times 3 = 0.8\cdot 0.2\cdot \frac{1}{0.169} \approx \boxed{0.947}$$

3 $$P(WE) = P(WE|A)P(A) + P(WE|B)P(B) + P(WE|C)P(C) + P(WE|D)P(D)$$

$$= 0.16 + 0.0054 + 0.0024 + 0.0006 = 0.169$$

$P(WE|A) = P(W|E,A)\,P(E|A) = 0.8 \cdot 0.5 = 0.4$
$P(WE|B) = P(W|E,B)\,P(E|B) = 0.06 \cdot 0.3 = 0.018$
$P(WE|C) = 0.06 \cdot 0.2 = 0.012$;
$P(WE|D) = 0.06 \cdot 0.1 = 0.0006$

(i) $P(W|A,E) = P(W|A) = 0.8$ since the identification probability does not depend on the vehicle type

Pt. (2) Prior probability

$$P(A|E) = \frac{P(A,E)}{P(E)} = \frac{\overset{0.5}{P(E|A)}\,\overset{0.4}{P(A)}}{0.34} = \frac{0.2}{0.34} \approx \boxed{0.588}$$

The witness testimony increases the chances that the taxi was from A

Pt. (3) The question asks to change the assumptions on the electric car proportions and fleet sizes to make $P(A|E,W) < P(A|E)$

$$\frac{P(A|E,W)}{P(A|E)} = \frac{P(W|A,E)}{P(W|E)} = \frac{0.8}{P(W|E)} < 1 \iff P(W|E) > 0.8$$

It is not possible to achieve this changing $P(A), \dots, P(D)$ and $P(E|A), \dots, P(E|D)$. The only way to make the witness's testimony be not helpful is to make the witness less accurate.

Problem 2. A game show places one black ball and n white balls into an urn. Players take turns drawing one ball at a time without replacement; the player who draws the black ball loses.

You are allowed to choose your position in the drawing order among $n+1$ players.

(1) Compute the probability that you lose if you draw in position k.
(2) Does this probability depend on k? Justify your answer.
(3) Now suppose that before each draw, the host randomly removes one of the remaining *white* balls (if any) and discards it. How does this modification affect your optimal choice of position? Clearly, every round removes 2 balls (or one, if this is the black ball, which is the last one to survive). Thus, if $k > \lfloor (n+1)/2 \rfloor$, you will not get a chance to play. Please analyze the choices $1 \le k \le \lfloor (n+1)/2 \rfloor$.

(1) Let $p_i = \mathbb{P}(\text{loss in position } k)$

$$p_1 = \frac{1}{n+1}$$

$$p_2 = \mathbb{P}(1^{st}\text{ white}, 2^{nd}\text{ black}) = \frac{n}{n+1}\cdot\frac{1}{n} = \frac{1}{n+1}$$

...

$$p_k = \frac{n}{n+1}\,\frac{n-1}{n}\,\frac{n-2}{n-1}\cdots\frac{n-(k-2)}{n-(k-3)}\,\frac{1}{n-(k-2)} = \frac{1}{n+1}$$

(2) By part (1) the probability of loss does not depend on k.
We can argue as follows: suppose there are $n+1$ boxes and player i always takes the ball from box i.
Before placing the balls in the boxes, the balls are randomly permuted. Since the black ball can be in any of the boxes with probability $\frac{1}{n+1}$, the loss chances for the players are equal.

(3) Here # balls left before player k draws a ball $= \overset{k=1}{n}, \overset{k=2}{n-2}, \ldots \overset{\text{general } k}{n-2k+2}$

$$p_1 = \frac{1}{n}, \quad p_2 = \frac{n-1}{n}\cdot\frac{1}{n-2}$$

$$p_k = \frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{n-2k+3}{n-2k+4}\,\frac{1}{n-2k+2}$$

$\therefore$ The loss probability increases with k

Go first!

$$\frac{p_{k+1}}{p_k} = \frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{n-2k+1}{n-2k+2}\,\frac{1}{n-2k}\cdot\frac{n(n-2)\cdots(n-2k+4)(n-2k+2)}{(n-1)(n-3)\cdots(n-2k+3)} = \frac{n-2k+1}{n-2k} > 1$$

as long as $k < \left\lfloor \frac{n+1}{2} \right\rfloor$

Problem 3. Two basketball players, Player 1 and Player 2, independently attempt free throws. Each shot is successful with probability p, independently of all other shots.

For $i = 1, 2$, let T_i be the trial on which Player i makes their *second* successful shot.

(1) Find $P(T_i = n)$ for $n \geq 2$.
(2) Find $P(\min(T_1, T_2) = n)$.
(3) What is the probability that Player 1 reaches two successes before Player 2?

(1) $P(T_i = n) = (n-1)\,p\,(1-p)^{n-2} \cdot p = (n-1)\,p^2\,(1-p)^{n-2}$

(2) $P(T_2 \geq n) = P(\leq 1 \text{ success in } n-1 \text{ throws})$

$= (1-p)^{n-1} + (n-1)\,p(1-p)^{n-2}$

$P(\min(T_1, T_2) = n) = 2P(T_1 = n, T_2 \geq n) = 2P(T_1 = n)\,P(T_2 \geq n)$

$= 2\,(n-1)\,p^2\,(1-p)^{n-2}\left((1-p)^{n-1} + (n-1)\,p(1-p)^{n-2}\right)$

(3) The game is symmetric, so

$$\boxed{\begin{gathered} P(T_1 < T_2) + P(T_2 > T_1) + P(T_1 = T_2) = 1 \\ P(T_1 < T_2) = \tfrac{1}{2}\left(1 - P(T_1 = T_2)\right) \end{gathered}}$$

Let us compute

$P(T_1 = T_2) = \sum_{n=2}^{\infty} \left((n-1)\,p^2\,(1-p)^{n-2}\right)^2 = p^4 \sum_{n=2}^{\infty} (n-1)^2\,(1-p)^{2n-4}$

$= p^4 \boxed{\sum_{k=0}^{\infty} (k+1)^2\left((1-p)^2\right)^k} = p^4\,\dfrac{1+(1-p)^2}{\left(1-(1-p)^2\right)^3} = \dfrac{p\,(1+(1-p)^2)}{(2-p)^3}$

see calculation below

$P(T_1 < T_2) = \frac{1}{2}\left(1 - P(T_1 = T_2)\right) = \frac{1}{2}\left(1 - \dfrac{p\,(1+(1-p)^2)}{(2-p)^3}\right)$ //

For instance, if $p = \frac{1}{2}$, $P(T_1 = T_2) = \frac{1}{9}$; $P(T_1 < T_2) = \frac{4}{9}$

A standard way to compute sums like the one in the box is as follows

start with the geometric sum $\sum_{k=0}^{\infty} s^k = \frac{1}{1-s} \quad (|s|<1)$

$$\frac{d}{ds}\sum_{k=0}^{\infty} s^k = \begin{cases} \frac{d}{ds}\frac{1}{1-s} = \frac{1}{(1-s)^2} \\ \sum_{k\geq 1} k s^{k-1} = \frac{1}{s}\sum_{k\geq 1} k s^k \end{cases} \qquad \therefore \sum_{k\geq 1} k s^k = \frac{s}{(1-s)^2}$$

$$\frac{d^2}{ds^2}\sum_{k\geq 1} s^k = \begin{cases} \frac{d}{ds}\frac{1}{(1-s)^2} = \frac{2}{(1-s)^3} \\ \frac{d}{ds}\sum_{k\geq 1} k s^{k-1} = \sum_{k\geq 1} k(k-1) s^{k-2} = \frac{1}{s^2}\Big(\underbrace{\sum_{k\geq 1} k^2 s^k}_{} - \underbrace{\sum k s^k}_{\text{know from above}}\Big) \end{cases}$$

$$\therefore \sum_{k\geq 1} k^2 s^k = \sum k s^k + \frac{2s^2}{(1-s)^3} = \frac{s(1+s)}{(1-s)^3}$$

Our task is to find $\sum_{k\geq 1} (k+1)^2 s^k$ where $s = (1-p)^2$

$$\sum_{k\geq 0} (k+1)^2 s^k = \frac{1}{s}\sum_{k\geq 1} k^2 s^k = \frac{1+s}{(1-s)^3} = \boxed{\frac{1+(1-p)^2}{(1-(1-p)^2)^3}}$$

Problem 4. A shipment of electronic components contains 8 components, of which 3 are defective. An inspector tests components one by one, chosen uniformly at random without replacement, until either two defectives are found or all components have been tested.

Let N be the number of components tested.

(1) Find $P(N = 2)$, $P(N = 3)$, and $P(N = 4)$.
(2) What is the probability that all 8 components are tested?
(3) What is the probability that at least five components are tested?

$$(1) \quad P(N=2) = \frac{\binom{3}{2}\binom{5}{0}}{\binom{8}{2}} = \frac{6}{56} = \frac{3}{28} \approx 0.107 \text{ //}$$

$$P(N=3) = \frac{\overbrace{\binom{3}{1}\binom{5}{2-1}}^{\text{1 def among first 2}}}{\binom{8}{2}} \cdot \overbrace{\frac{2}{6}}^{3^{rd}\text{ def}} = \frac{15}{4 \cdot 7} \cdot \frac{1}{3} = \frac{5}{28} \approx 0.179 \text{ //}$$

$$P(N=4) = \frac{\binom{3}{1}\binom{5}{3-1}}{\binom{8}{3}} \cdot \frac{2}{5} = \frac{3 \cdot 10}{8 \cdot 7} \cdot \frac{2}{5} = \frac{3}{14} \approx 0.214 \text{ //}$$

(2) $P(N=8) = 0$ b/c. the first 7 must contain ≥ 2 defective

(3) $P(N \geq 5) = P(\leq 1 \text{ def among the first 4 tested})$

$$= \frac{\binom{3}{1}\binom{5}{3} + \binom{3}{0}\binom{5}{4}}{\binom{8}{4}} = \frac{35}{70} = \frac{1}{2}$$

Problem 5. A fair die is rolled repeatedly until the first time a 6 appears. Let N be the total number of rolls.

(1) Find the pmf of N.
(2) What is $P(N \text{ is even})$?
(3) Given that $N \geq 3$, what is $P(N = 3)$?
(4) Let X be the number of 1's observed before the first 6. Find the pmf of X. Hint: Ignore all rolls that are neither 1 nor 6. These rolls do not affect which of 1 or 6 appears first.

(1) $N \sim \text{Geom}\left(\frac{1}{6}\right) + 1$ (First success RV)

$$p_N(n) = (1-p)^{n-1} p, \quad n = 1, 2, \ldots \quad \text{where } p = \frac{1}{6}$$

(2)
$$P(N \text{ is even}) = p \sum_{n=2}^{\infty} (1-p)^{2n-3} = \frac{p}{(1-p)^3} \frac{(1-p)^4}{1-(1-p)^2} = \frac{1-p}{2-p} = \frac{5/6}{11/6} = \frac{5}{11}$$

(3)
$$P(N=3 \mid N \geq 3) = \frac{P(N=3)}{P(N \geq 3)} = \frac{(1-p)^2 p}{1-p-(1-p)p} = \frac{(1-p)^2 p}{(1-p)^2} = p \quad \left(= \frac{1}{6}\right)$$

This is clear without counting: we are given that $N=3$ was reached; our goal is to stop there, and this happens with prob. $\frac{1}{6}$

(4) Following the hint, forget the outcomes other than 1 or 6. We live on the space $\{1, 6\}$; they are equiprobable, i.e., given that 1 or 6 appears, $P(1) = P(6) = 1/2$

$$p_X(k) = 2^{-k-1}, \quad k = 0, 1, 2, \ldots$$

Problem 6. A box contains 5 green balls and 7 yellow balls. Balls are drawn one at a time without replacement until all green balls have been drawn. Let T be the number of draws required.

(1) What are the possible values of T?
(2) For $k = 5, 6, \ldots, 12$, express $P(T = k)$ in terms of binomial coefficients.
(3) Compute $P(T = 12)$.
(4) Given that exactly 3 yellow balls were drawn among the first 6 draws, what is the probability that $T = 9$?

(1) Range $(T) = 5, 6, .., 12$ $\quad\quad 12-k$

(2) It would be incorrect to use hypergeometric b/c. it does not account for the requirement that k^{th} draw is green

$$P(T=k) = P(k-1 \text{ draws contain 4 green balls} ; k^{th} \text{ draw green})$$
$$= P(k-1 \text{ draws with 4 green}) P(k^{th} \mid 4 \text{ green in } k-1 \text{ draws})$$

$$= \underbrace{\frac{\binom{5}{4}\binom{7}{k-5}}{\binom{12}{k-1}}}_{\text{Using hypergeometric}} \cdot \frac{1}{12-(k-1)}$$

$=$ # of balls left after $k-1$ draws

(3) $P(T=12) = \dfrac{\binom{5}{4}\binom{7}{7}}{\binom{12}{11}} = \dfrac{5}{12}$ //

(4) We are left with 4 yellow and 2 green balls. let $T' = T-6$

$$P(T=9) = P(T'=3) = \frac{\binom{2}{1}\binom{4}{3-2}}{\binom{6}{2}} \frac{1}{4} = \frac{2 \cdot 4}{15 \cdot 4} = \frac{2}{15} //$$