

**ENEE324, Home assignment 3. Date due September 29, 2026, 11:59pm EST.**

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points.

**Problem 1.** Two identical-looking bags sit on a table. Bag 1 contains 3 red and 7 green balls; Bag 2 contains 6 red and 4 green balls. You pick one of the two bags at random (each with probability  $1/2$ ) and then draw balls from that same bag *without replacement*. The first two balls you draw are both red.

- (a) What is the probability that you are drawing from Bag 1?
- (b) Given the two red balls already observed, what is the probability that the *next* (third) ball drawn from the same bag is also red?

**Problem 2.** Urn  $A$  contains 2 white and 3 black balls; urn  $B$  contains 3 white and 5 black balls. One ball is drawn at random from  $A$  and transferred (without looking at its color) into  $B$ . A ball is then drawn at random from  $B$ .

- (a) What is the probability that the ball drawn from  $B$  is white?
- (b) Given that the ball drawn from  $B$  is white, what is the probability that the transferred ball was white?
- (c) Let  $T$  be the event "the transferred ball is white" and  $W$  the event "the ball drawn from  $B$  is white". Are  $T$  and  $W$  independent?

**Problem 3.** A box contains three cards. One card is red on both sides (R/R), one is black on both sides (B/B), and one is red on one side and black on the other (R/B). A card is chosen at random and laid on the table; the side that shows is red.

- (a) Given that the visible side is red, what is the probability that the hidden side is also red?
- (b) A common but incorrect answer to (a) is  $1/2$  ("the card is either R/R or R/B, so it is equally likely to be either"). Explain precisely what is wrong with this reasoning.
- (c) Now suppose the box instead contains *two* R/B cards and *one* R/R card (still three cards). Again a card is drawn and the visible side is red. What is the probability that the hidden side is red?

**Problem 4.** An urn contains 6 white and 4 black balls. Three balls are drawn one at a time, without replacement.

- (a) What is the probability that the *second* ball drawn is black?
- (b) What is the probability that the first two balls are white and the third is black?
- (c) Given that the third ball drawn is black, what is the probability that the first two balls are both white?
- (d) Suppose instead the three draws are made *with* replacement. Recompute the probabilities in parts (a) and (c) under this model, and compare.

**Problem 5.** A bit  $X \in \{0, 1\}$  with  $P(X = 1) = 0.4$  is sent through a relay consisting of two binary symmetric channels in series. Each channel, independently, flips the bit passing through it with probability  $\varepsilon = 0.2$ . Let  $Y$  be the output of the first channel (the relay's intermediate value) and  $Z$  the final output.

- (a) Find the end-to-end error probability  $P(Z \neq X)$ .
- (b) Given that the receiver observes  $Z = 1$ , find the posterior probability  $P(X = 1 \mid Z = 1)$ .
- (c) Are  $X$  and  $Z$  conditionally independent given  $Y$ ? Justify your answer.

**Problem 6.** A short quiz has three questions, worth 1, 2 and 3 points. A student answers them correctly, independently of one another, with probabilities 0.8, 0.5 and 0.3 respectively. Let  $X$  be the student's total score.

- (a) Find the range of  $X$  and compute the pmf  $p_X(x)$ , indicating which outcomes of the experiment contribute to each value. Verify that  $\sum_x p_X(x) = 1$ .
- (b) Passing requires at least 3 points. Find  $P(X \geq 3)$ .
- (c) Write the CDF  $F_X(x)$  for all real  $x$ , and use it to compute  $P(1 < X \leq 4)$ .

**Problem 7.** Four data bits are encoded into a block of  $n = 7$  bits and these 7 bits are sent over a noisy link one by one. Each of the 7 bits is flipped, independently of the others, with probability  $\varepsilon$ . The encoding is constructed in a way that allows to recover all four data bits if and only if at most one bit of the block of 7 was flipped. Let  $X$  be the number of flipped bits in the block.

- (a) Which random variable does  $X$  represent and what is the distribution of  $X$ ? Write down  $p_X(k)$  and justify it.
- (b) Take  $\varepsilon = 0.1$  and compute  $P(X = 0)$ ,  $P(X = 1)$  and  $P(X \geq 2)$ .
- (c) Still at  $\varepsilon = 0.1$ , find the probability that the four data bits are recovered, and compare it with sending the four data bits over the same channel with no coding at all — which succeeds only if none of those four bits is flipped.
- (d) Show that for every  $\varepsilon$  the ratio of the two success probabilities in (c) equals  $(1 - \varepsilon)^2(1 + 6\varepsilon)$ . Deduce that the code beats no coding for every  $0 < \varepsilon < \frac{1}{2}$ , with equality at  $\varepsilon = \frac{1}{2}$ , and explain what is special about  $\varepsilon = \frac{1}{2}$ .
- (e) Simulate the transmission using the following encoding: the data bits  $x_1, x_2, x_3, x_4$  are transmitted as a 7-bit block  $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ , where

$$x_5 = x_1 \oplus x_2 \oplus x_4, \quad x_6 = x_1 \oplus x_3 \oplus x_4, \quad x_7 = x_2 \oplus x_3 \oplus x_4$$

and where  $\oplus$  is the XOR operation. Your task is to approximate the probability of incorrect decoding, i.e., the probability of the event that the string received on the output of the communication link differs from  $\mathbf{x}$  by more than one bit. Perform  $10^2; 10^3; 10^5$  simulation rounds and record the probability values. How many rounds did you need to approximate the exact value of the probability of failed data recovery from the encoded block to within  $10^{-3}$ ?