ENEE324, Home assignment 4. Date due October 13, 2025, 11:59pm EDT.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.

- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. Let X be a random variable that takes values $\{1, 2, 3, 4\}$ with probabilities $\{0.1, 0.3, 0.4, 0.2\}$. Define a random variable $Y = 5 - X$.

1. Find the PMF of Y.
2. Compute the expectation $E(X + Y)$.
3. Are X and Y independent? Justification required.

1. Range $(Y) = \{1,2,3,4\}$

$p_Y(x) = p_X(5-x)$

x	1	2	3	4
$p_Y(x)$	0.2	0.4	0.3	0.1

2. $EY = \sum_{x=1}^{4} x p_Y(x) = \sum_{x=1}^{4} x p_X(5-x) = \sum_{x=1}^{4} (5-x) p_X(x) = 5 - EX$

$E(X+Y) = 5$

Or, compute directly:

$EX = 0.1 + 2\cdot 0.3 + 3 \cdot 0.4 + 4 \cdot 0.2 = 0.1 + 0.6 + 1.2 + 0.8 = 2.7$

$EY = 0.2 + 0.8 + 0.9 + 0.4 = 2.3$; $E(X+Y) = EX + EY = 5$

3. X and Y are not independent:

$P(X=1; Y=4) = P(X=1) = 0.1$

$P(X=1) P(Y=4) = (P(X=1))^2 = 0.01$

Problem 2. A factory produces microchips of 2 types, 70% of type A and 30% of type B. Chips of type A last a random number X_A of years with pmf $p_{X_A}(k) = \frac{1}{4}(\frac{3}{4})^{k-1}, k = 1, 2, \ldots$, and chips of type B last X_B years with pmf $p_{X_B}(k) = \frac{1}{2}(\frac{1}{2})^{k-1}, k = 1, 2, \ldots$.

1. Verify that $p_{X_A}(k)$ and $p_{X_b}(k)$ are valid pmf's. Justification required.

A chip is chosen at random, ~~so it is of type A or B with probability $\frac{1}{2}$ each~~. Let X denote its random lifetime.

2. Find the pmf of X.
3. Compute the expectation $E(X)$.
4. Compute $P(X \geq 4)$.

1. $\sum_{k=1}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^{k-1} = \frac{1}{4} \cdot \frac{1}{1-3/4} = 1.$ $\qquad \sum_{k=1}^{\infty} \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{k-1} = \frac{1}{2} \cdot \frac{1}{1-1/2} = 1$

2. $P(A) = 0.7$, $P(B) = 0.3$. As in Part 4 below, use LOTP

$$P(X=k) = P(X=k|A)\,P(A) + P(X=k|B)\,P(B) = \frac{1}{4}\cdot\left(\frac{3}{4}\right)^{k-1} \cdot 0.7 + \frac{1}{2}\left(\frac{1}{2}\right)^{k-1} \cdot 0.3$$

$$= 0.175 \cdot \frac{3}{4}^{k-1} + 0.15 \cdot \frac{1}{2}^{k-1}, \quad k = 1, 2, \ldots$$

3. X_A = FirstSuccess random variable ; $EX_A = \frac{1}{p} = 4$

Similarly, $EX_B = \frac{1}{2}$

X is a mixture of X_A and X_B

$\therefore EX = P(A)\,EX_A + P(B)\,EX_B = 0.7 \cdot 4 + 0.3 \cdot 2 = 3.4$

4. Using law of total probability :

$$P(X \geq 4) = \underset{P(X \geq 4 | A)}{\underset{\parallel}{P(X_A \geq 4)}}\,P(A) + \underset{P(X \geq 4 | B)}{\underset{\parallel}{P(X_B \geq 4)}}\,P(B) = 0.7 \cdot \left(\frac{3}{4}\right)^3 + 0.3 \cdot \left(\frac{1}{2}\right)^3$$

Computed in Lecture 8, p.2 for geometric RVs. Our case is first success = geom + 1.

$$= 0.7 \cdot \frac{27}{64} + 0.3 \cdot \frac{1}{8} = 0.295 + 0.0375$$

$$= 0.3325$$

If you solved this problem with $P(A) = P(B) = \frac{1}{2}$, that's fine, too

Problem 3. There are 7 levels in a video game. You start at level 1 and have probability p_1 of advancing to level 2 (so with probability $1 - p_1$ your game ends at level 1, without ever advancing). If you reach level j, your probability of advancing to level $j+1$ is p_j, for all $j = 1, 2, \ldots, 6$. Denote by X the number of the highest level that you reach.

1. Find the pmf of X in terms of the numbers p_j.
2. Letting $p_j = 0.8 - 0.1j, j = 1, 2, \ldots, 6$, find EX, the expected level to which you advance.

1. $P(X=k) = \left(\prod_{i=1}^{k-1} p_i\right)(1-p_k)$, $k = 1, 2, \ldots, 6$; $\quad P(X=7) = \prod_{i=1}^{6} p_i$

For $k=1$, the product is empty, so $P(X=1) = p_1$

2. $p_1 = 0.7$, $p_2 = 0.6$, $\ldots$, $p_6 = 0.2$

$$EX = \sum_{k=1}^{7} k \, P(X=k) = \sum_{k=1}^{6} k \prod_{i=1}^{k-1} p_i (1-p_k) + 7 \prod_{i=1}^{6} p_i$$

k	1	2	3	4	5	6	7
$p_X(k)$	0.3	0.28	0.21	0.126	0.059	0.021	0.005

$EX = 2.44$

Problem 4. In a lottery, the probability for each ticket to be winning is p. You hope that buying 3 tickets will triple your chances of having at least one winning ticket.

1. What is the distribution of the number of winning tickets out of the three tickets you bought ? (find the pmf of the number X of winning tickets)

2. Find the probability that at least one of the three tickets is winning. Obtain the answer in two different ways:

   2a. Use the inclusion-exclusion formula.

   2b. Start with computing the probability of the complementary event.

3. By now you are convinced that your hope is not going to materialize. For which values of p your belief of tripling your chances is validated with no more than 1% margin of error? (hint: for small p).

1. $X \sim Bin(3,p)$

$$p_X(k) = \binom{3}{k} p^k (1-p)^{n-k}, \quad k=0,1,2,3$$

2.

2a. Let A_i, $i=1,2,3$ be the event $\{i^{th}$ ticket winning$\}$

$$P(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + P(A_1 \cap A_2 \cap A_3)$$

$$= 3p - 3p^2 + p^3$$

2b. $$P((A_1 \cup A_2 \cup A_3)^c) = P\left(\bigcap_{i=1}^{3} A_i^c\right) = (1-p)^3$$

$$P(A_1 \cup A_2 \cup A_3) = 1 - (1-p)^3 = 3p - 3p^2 + p^3$$

3. The difference is $3p - P(A_1 \cup A_2 \cup A_3) = p^2(3-p)$

It is ≤ 0.01 for $0 \leq p < 0.058$

Problem 5. (a) Let X be a random month of the year, numbered from 1 to 12, and let Y be the month after that, also represented as a number. Do X and Y have the same distribution? What is $P(X < Y)$?

(b) Suppose that two discrete RVs X and Y have the same distribution, but $P(X < Y) \geq p$. Is it possible that

- $p = 0.99$?
- $p = 0.999$?
- $p = 1$?

In each case give an example of random variables X, Y with $P(X < Y) \geq p$ or show that it is impossible.

(a) Yes, the distribution of X and Y is the same:

$$p(k) = \frac{1}{12} \text{ for all } 1 \leq k \leq 12$$

$$P(X<Y) = \frac{11}{12} \text{ because } X<Y \text{ unless } X=12$$

(b) If $p=0.99$ or 0.999, it is possible: just suppose that there are 100 (or 1000) months

However, for $p=1$ RVs X and Y with the same distribution that satisfy $P(X<Y)=1$ do not exist.

For instance, if Both EX and $EY < \infty$, then $E[Y-X]=EY-EX=0$ since X and Y have the same distribution.

At the same time, the random variable $Z=Y-X$ satisfies $P(Z>0)=1$, so $EZ = \sum_z z\,P(Z=z) > 0$.

A similar idea works even for X and Y with infinite expectations.

Problem 6. In the home assignments for some imagined class there were different 100 problems. 10 problems out of this list, randomly selected, will appear on the exam. You pass if you solve correctly 7 problems out of the 10 included in the exam paper. You decide to boost your chances by memorizing solutions of s problems out of the entire set of 100, where $0 \leq s \leq 100$.

(a) Let X be the number of problems appearing in the midterm paper that you have memorized. What is the distribution of X? Identify it as one of the named distributions studied in class and give the pmf $p_X(k)$ in terms of s.

(b) Find the smallest number s such that your chance of success is at least 50%.

(a) $X \sim$ Hypergeometric $(10, 90, s)$

where s = # of problems that you have memorized. Let m = # of prepared problems in the exam paper, then

$$p_X(m) = \frac{\binom{10}{m}\binom{90}{s-m}}{\binom{100}{s}} = \frac{\binom{s}{m}\binom{100-s}{10-m}}{\binom{100}{10}}, \quad m = 0, 1, \dots, 10$$

(b) Find the smallest s s.t.

$$\sum_{m=7}^{10} p_X(m) > 0.5$$

Answer: $s = 65$ (found by computer)