ENEE324, Home assignment 4. Date due March 25, 2026, 11:59pm EDT.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

• Submissions on paper or by email will not be accepted.
• Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.

• Justification of solutions is required.
• Each problem is worth 10 points unless noted otherwise.

Problem 1. Let X be a discrete random variable taking values in $\{0, 1, 2, 3\}$ with

$$P(X = k) = \frac{k+1}{10}, \qquad k = 0, 1, 2, 3.$$

(1) Verify that this defines a probability distribution.
(2) Compute $\mathbb{E}[X]$.
(3) Compute $\mathbb{E}[X^2]$.

(1) $\sum_{k=0}^{3} (k+1) = 1+2+3+4 = 10 \quad \therefore \quad \sum P(X=k) = 1$

(2) $\mathbb{E}X = \sum_{k=1,2,3} k\, P(X=1) = 1\cdot 0.2 + 2\cdot 0.3 + 3\cdot 0.4 = 0.2 + 0.6 + 1.2 = 2$

(3) $\mathbb{E}X^2 = 1\cdot 0.2 + 4\cdot 0.3 + 9\cdot 0.4 = 0.2 + 1.2 + 3.6 = 5$

Problem 2. Let $X \sim \mathrm{FS}(p)$ and $Y \sim \mathrm{FS}(p)$ be independent First Success (=geometric+1) random variables with

$$P(X = k) = P(Y = k) = p(1-p)^{k-1}, \qquad k = 1, 2, \ldots.$$

Define

$$U = \min(X, Y), \qquad V = X - Y.$$

(1) Show that X and Y are identically distributed and compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
(2) Show that U is an FS random variable and find its parameter.
(3) Compute $P(V = 0)$ and $P(V = 1)$.
(4) Compute $\mathbb{E}[\mathbf{1}_{\{X<Y\}}]$.
(5) Are U and V independent? Justify your answer.

(1) X and Y are identically distributed by their definition

$$\mathbb{E}X = \mathbb{E}Y = \frac{1}{p}$$

(2) $P(U \leq n) = P(\min(X,Y) \leq n) = 1 - P(\min(X,Y) > n)$

$= 1 - P(X>n)\,P(Y>n)$

Use CDF of 1^{st} success RV,

$$F_X(n) = P(X \leq n) = \sum_{k=0}^{n-1} p(1-p)^k = p\,\frac{1-(1-p)^n}{p} = 1-(1-p)^n$$

$\therefore\ G_X(n) = (1-p)^n$
survival function

$$P(U \leq n) = 1-(1-p)^{2n} = 1-\left((1-p)^2\right)^n$$

$$(1-p)^2 = 1 - \underbrace{p(2-p)}$$

$\therefore$ U is $FS(p(2-p))$

(3) $$P(X-Y=0) = \sum_{k=1}^{\infty} P(X=k)P(Y=k) = \sum_{k=1} p^2(1-p)^{2(k-1)}$$

$$= p^2\,\frac{1}{1-(1-p)^2} = \frac{p}{2-p}$$

$$P(X-Y=1) = \sum_{k=1} P(X=k)\,P(Y=k-1) = \sum_{k=2}^{\infty} p^2(1-p)^{k-1}(1-p)^{k-2}$$

$$= \frac{p^2}{(1-p)^3}\sum_{k\geq 2}(1-p)^{2k} = \frac{p^2}{(1-p)^3}\,\frac{(1-p)^4}{(2-p)p} = \frac{p(1-p)}{2-p}$$

(4) $P(U=1 \mid V=1) = P(\min(X,Y)=1 \mid X-Y=1)$

$= P(X=2, Y=1) = p^2(1-p)$

Since $U \sim FS(p(2-p))$, $P(U=1) = p(2-p)$

Since $P(U=1|V=1) \neq P(U=1)$, U and V are dependent

Problem 3. A city has n parking spots along a street and n drivers arrive one after another. Each driver independently chooses one of the n spots uniformly at random and parks there if the spot is empty; otherwise the driver leaves.

Let X be the number of occupied parking spots after all n drivers have arrived.

(1) For $i = 1, \ldots, n$, define an indicator random variable I_i for the event that spot i is occupied. Express X in terms of $I_1, \ldots, I_n$.
(2) Compute $\mathbb{E}[I_i]$.
(3) Use linearity of expectation to compute $\mathbb{E}[X]$.

1. $X = \sum_{i=1}^{n} I_i$

2. Let $S_j^{(i)} = \{ j^{th} \text{ driver chooses spot } i \}$

$$\mathbb{E}I_i = P(I_i = 1) = P(S_1^{(i)} \cup S_2^{(i)} \cup \ldots \cup S_n^{(i)})$$

$$= 1 - P((S_1^{(i)})^c \cap (S_2^{(i)})^c \cap \ldots \cap (S_n^{(i)})^c)$$

$$= 1 - P(\text{nobody chooses } i^{th} \text{ spot}) = 1 - \left(1 - \frac{1}{n}\right)^n$$

3. $\mathbb{E}X = n\left(1 - \left(1 - \frac{1}{n}\right)^n\right)$

Recalling that $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$,

$$\mathbb{E}X \approx n\left(1 - \frac{1}{e}\right)$$

Problem 4. Let X be a discrete random variable with distribution
$$P(X=k)=\frac{6}{\pi^2}\frac{1}{k^2}, \qquad k=1,2,3,\ldots.$$

(1) Verify that this defines a probability distribution.
(2) Determine whether $\mathbb{E}[X]$ is finite.
(3) Let $Y=\frac{1}{X}$. Compute $\mathbb{E}[Y]$.
(4) Let $Z=\mathbf{1}_{\{X \text{ is even}\}}$. Compute $\mathbb{E}[Z]$.

(1) It is well known that $\sum_{k\geq 1}\frac{1}{k^2}=\frac{\pi^2}{6}$

This can be shown in many different ways. For instance, take

$$f(x)=(\pi-x)^2, \quad 0\leq x\leq 2\pi,$$

extended periodically. Its Fourier series is easily found to be

$$f(x)=\frac{\pi^2}{3}+\sum_{k=1}^{\infty}\frac{4}{k^2}\cos kx.$$

Take $x=0$: $\pi^2=\frac{\pi^2}{3}+\sum_{k=1}^{\infty}\frac{4}{k^2}$, or $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$

This shows that $\sum_{k=1}^{\infty}P(X=k)=1$

(2) $$\mathbb{E}X=\sum_{k=1}^{\infty}k\,P(X=k)=\frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k}=\infty$$

Ans. $\mathbb{E}X$ is infinite

(3) Using LOTUS,

$$\mathbb{E}Y=\frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^3}\approx\frac{6}{\pi^2}\cdot 1.202\approx 0.7307$$

(4) $$EZ=P(X\text{ even})=\sum_{k=1}^{\infty}\frac{6}{\pi^2}\frac{1}{(2k)^2}=\frac{6}{4\pi^2}\cdot\frac{\pi^2}{6}=\frac{1}{4}$$

Problem 5. n balls are thrown independently and uniformly into m bins. Let X be the number of empty bins.

(1) Define indicator variables for the event that bin i is empty.
(2) Express X as a sum of indicators.
(3) Compute $\mathbb{E}[X]$.

(1) Let $I_i = \mathbb{1}(\text{bin } i \text{ empty})$

$$X = \sum_{i=1}^{m} I_i$$

(2) $P(I_i = 1) = \left(1 - \frac{1}{m}\right)^n$

(3) $\mathbb{E}X = \sum_{i=1}^{m} \mathbb{E} I_i = m \left(1 - \frac{1}{m}\right)^n$

Problem 6. Two fair dice are rolled. Let X be the result of the first die and let

$$Y = \begin{cases} 1, & \text{if } X \text{ is even,} \\ 0, & \text{if } X \text{ is odd.} \end{cases}$$

(1) Find the distributions of X and Y.
(2) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
(3) Are X and Y independent? Justify your answer.

(1) $p_X(i) = \frac{1}{6}$, $i = 1, \dots, 6$

$p_Y(i) = \frac{1}{2}$, $i = 0, 1$

(2) $\mathbb{E}X = 3.5$; $\mathbb{E}Y = \frac{1}{2}$

(3) $P(X = 1 \cap Y = 1) = 0$

$P(X=1)\,P(Y=1) = \frac{1}{12}$

$\therefore$ X and Y are not independent