ENEE324, Home assignment 4. Date due March 25, 2026, 11:59pm EDT.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.

- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. Let X be a discrete random variable taking values in $\{0, 1, 2, 3\}$ with

$$P(X = k) = \frac{k+1}{10}, \qquad k = 0, 1, 2, 3.$$

(1) Verify that this defines a probability distribution.
(2) Compute $\mathbb{E}[X]$.
(3) Compute $\mathbb{E}[X^2]$.

Problem 2. Let $X \sim \text{FS}(p)$ and $Y \sim \text{FS}(p)$ be independent First Success (=geometric+1) random variables with

$$P(X = k) = P(Y = k) = p(1-p)^{k-1}, \qquad k = 1, 2, \dots.$$

Define

$$U = \min(X, Y), \qquad V = X - Y.$$

(1) Show that X and Y are identically distributed and compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
(2) Show that U is an FS random variable and find its parameter.
(3) Compute $P(V = 0)$ and $P(V = 1)$.
(4) Compute $\mathbb{E}[\mathbf{1}_{\{X<Y\}}]$.
(5) Are U and V independent? Justify your answer.

Problem 3. A city has n parking spots along a street and n drivers arrive one after another. Each driver independently chooses one of the n spots uniformly at random and parks there if the spot is empty; otherwise the driver leaves.

Let X be the number of occupied parking spots after all n drivers have arrived.

(1) For $i = 1, \dots, n$, define an indicator random variable I_i for the event that spot i is occupied. Express X in terms of $I_1, \dots, I_n$.
(2) Compute $\mathbb{E}[I_i]$.
(3) Use linearity of expectation to compute $\mathbb{E}[X]$.

Problem 4. Let X be a discrete random variable with distribution

$$P(X = k) = \frac{6}{\pi^2}\frac{1}{k^2}, \qquad k = 1, 2, 3, \dots.$$

(1) Verify that this defines a probability distribution.
(2) Determine whether $\mathbb{E}[X]$ is finite.
(3) Let $Y = \frac{1}{X}$. Compute $\mathbb{E}[Y]$.
(4) Let $Z = \mathbf{1}_{\{X \text{ is even}\}}$. Compute $\mathbb{E}[Z]$.

Problem 5. n balls are thrown independently and uniformly into m bins. Let X be the number of empty bins.

(1) Define indicator variables for the event that bin i is empty.
(2) Express X as a sum of indicators.
(3) Compute $\mathbb{E}[X]$.

Problem 6. Two fair dice are rolled. Let X be the result of the first die and let

$$Y = \begin{cases} 1, & \text{if } X \text{ is even,} \\ 0, & \text{if } X \text{ is odd.} \end{cases}$$

(1) Find the distributions of X and Y.
(2) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.
(3) Are X and Y independent? Justify your answer.