ENEE324, Home assignment 5. Date due November 1, 2025, 11:59pm EDT.

Instructor: Alexander Barg

Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.

- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. A lottery ticket costs \$1. With probability $1/1000$ you win \$500; otherwise, you win nothing. Let X be your *net gain* (winnings minus the \$1 cost).

(a) Write the probability mass function of X.
(b) Compute $\mathbb{E}[X]$. Interpret the result in plain language (on average, do you gain or lose?).
(c) Compute $\mathrm{Var}(X)$ and the standard deviation.
(d) Suppose instead you buy $n = 1000$ independent tickets. Find $\mathbb{E}[S_n]$ and $\mathrm{Var}(S_n)$, where S_n is your total net gain.
(e) Compute the expected net gain per ticket in that case, and discuss whether buying many tickets changes your odds of profit.
(f) If the prize increases to \$1000 but the cost stays \$1, does the game become "fair"? Explain.

(a) $$p_X(k) = \begin{cases} 0.001 & k = 499 \\ -1 & \text{o/w} \end{cases}$$

(b) $EX = 0.499 - 0.999 = -0.5$; on average, you lose

(c) $EX^2 = 0.999 \cdot (-1)^2 + 0.001 \cdot 499^2 = 0.999 + (500-1)^2 \cdot 0.001$
$= 0.999 + 250 - 1 + 0.001 = 250$
$Var(X) = 250 - 0.25 = 249.75$; $\sigma_X =$

(d) # wins = Binom (1000, 0.001)
Let Y_m = net gain (loss) for m^{th} ticket, $m = 1, 2, \dots, 1000$
$EY_m = -0.5$ per Pt. (b)
$S_n = \sum_{m=1}^{1000} Y_m$; $ES_n = 1000\, EY_1 = -500$ by linearity

(e) Expected gain $= EY_m$; Buying many tickes $\Rightarrow$ greater loss

(f) In this case $EX = (1000-1) \cdot 0.001 - 0.999 = 0$, so fair

Problem 2. You roll two fair six-sided dice. Define

$$X = \begin{cases} 1, & \text{if the first die shows a higher number,} \\ 0, & \text{if the dice are equal,} \\ -1, & \text{if the second die shows a higher number.} \end{cases}$$

(a) Write the pmf of X by counting the favorable outcomes for each case.
(b) Compute $\mathbb{E}[X]$. (Is the game fair between the two dice?)
(c) Compute $\mathrm{Var}(X)$.
(d) Let $Y = 3X - 2$. Find $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.
(e) Suppose you play this dice game 100 times independently and let $T = \sum_{i=1}^{100} X_i$. Compute $\mathbb{E}[T]$ and $\mathrm{Var}(T)$.

(a) $P(1^{st} > 2^{nd}) = (5+4+3+2+1)\cdot\frac{1}{36} = \frac{15}{36} = P(1^{st} < 2^{nd})$ by symmetry

$P(1^{st} = 2^{nd}) = \frac{1}{6}$

$p_X(k) = \begin{cases} 5/12 & \text{if } k = 1 \text{ or } -1 \\ 1/6 & \text{if } k = 0 \end{cases}$

(b) $EX = 0$ by symmetry; the game is fair

(c) $EX^2 = \frac{10}{12}$; $Var(X) = EX^2 - 0 = \frac{5}{6}$

(d) $EY = 3EX - 2 = -2$ by linearity

$p_Y(k) = \begin{cases} 5/12 & k = 1 \\ 1/6 & k = 0 \\ 5/12 & k = -5 \end{cases}$

$Var(Y) = Var(3X-2) = 9\, Var X = \frac{15}{2} = 7.5$

(e) $ET = 100\, EX = 0$

$Var(T) \underset{\text{I}}{=} 100\, Var(X) = \frac{500}{6} = \frac{250}{3} = 83.33..$

by independence

Problem 3. On any given day, an online retailer may run a promotion. Let

$$X = \begin{cases} 1, & \text{promotion day,} \\ 0, & \text{ordinary day.} \end{cases}$$

Assume $P(X = 1) = 0.2$.

Let

$$Y = \begin{cases} 1, & \text{order delayed,} \\ 0, & \text{on-time delivery.} \end{cases}$$

Overall, the probability that a random order is delayed is $P(Y = 1) = 0.13$. During promotions, delays are three times as likely:

$$P(Y = 1 \mid X = 1) = 3\,P(Y = 1 \mid X = 0).$$

(a) Express $P(Y = 1)$ in terms of $P(X = 1)$, $P(X = 0)$, and the conditional probabilities.
(b) Use the given information to find $P(Y = 1 \mid X = 0)$ and $P(Y = 1 \mid X = 1)$.
(c) Compute the joint pmf $P(X = x, Y = y)$ for all $(x, y) \in \{0, 1\}^2$.
(d) Compute the marginal pmfs of X and Y.
(e) Are X and Y independent? Justify your answer.
(f) If improved logistics make promotion-day delays only twice as likely as normal, recompute $P(Y = 1)$ and argue whether the dependence conclusion changes.

(a) $P(Y=1) \underbrace{=}_{\text{LOTP}} P(Y=1|X=1)\,P(X=1) + P(Y=1|X=0)\,P(X=0)$

(b) We have $P(Y=1) = 0.13$, so

$0.13 = 3P(Y=1|X=0)\cdot 0.2 + P(Y=1|X=0)\cdot 0.8 = 1.4\,P(Y=1|X=0)$

$\therefore P(Y=1|X=0) = \frac{0.13}{1.4} \approx 0.093$; $P(Y=1|X=1) = 3\cdot 0.093 = 0.279$

Important: it is *NOT* true that $P(Y=1|X=0) + P(Y=1|X=1)$ "=" 1

(c)

$$P(X=x, Y=y) = \begin{cases} 0.8\cdot(\overbrace{1-0.093}^{0.907}) = 0.7256 & 0\ 0 \\ 0.2(\overbrace{1-0.279}^{0.721}) = 0.1442 & 1\ 0 \\ 0.8\cdot 0.093 = 0.0744 & 0\ 1 \\ 0.2\cdot 0.279 = 0.0558 & 1\ 1 \end{cases} \quad x, y$$

$p_X(1) = 0.2$; $p_X(0) = 0.8$

$p_Y(1) = 0.13$; $p_Y(0) = 0.87$

(d) $P(X=1, Y=1) = 0.0558$ $\quad P(X=1)P(Y=1) = 0.026 \Rightarrow$ not independent

(e) Let us recompute $P(Y=1)$: $P(Y=1|X=1) = 2\,\frac{0.13}{1.4} = \frac{13}{70} \approx 0.186$

$P(Y=1) = 0.2\cdot 0.186 + 0.8\cdot 0.093 \approx 0.11$

$P(X=1, Y=1) = P(Y=1|X=1)\,P(X=1) = 0.186\cdot 0.2 = 0.0372 \neq 0.2\cdot 0.11$

so still not independent

Problem 4. Calls to a customer–service center arrive randomly at an average rate of 5 calls per hour. Let X denote the number of calls received in one hour. Assume $X \sim \text{Poisson}(\lambda = 5)$.

(a) Compute $P(X = 3)$ and $P(X \leq 2)$.
(b) Find the expected number of calls and the variance.
(c) What is the probability of receiving at least one call during a 15–minute period?
(d) Suppose that each call requires on average 8 minutes of attention (independently of others). Find the expected total service time per hour. (Assume we count the full duration of each call that arrives within the hour.)
(e) Suppose each call requires on average 8 minutes of attention, and there are two agents working the whole hour. Each agent can serve on average $60/8 = 7.5$ calls, so together they can serve about 15 calls in one hour. What is the probability that the number of calls within the hour exceeds 15, so that the two agents cannot finish all calls within the hour on average?

(a) $P_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}$

$P(X=3) = \frac{5^3}{6} e^{-5} \approx 0.140$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = e^{-5}\left(1 + 5 + \frac{25}{2}\right) \approx 0.125$

(b) $EX = Var(X) = 5$ (for Poisson, $E = Var$)

(c) $15 \text{min} = \frac{1}{4} \text{hr}$ $\quad \lambda = 5 \cdot 0.25 = 1.25$

$P(\geq 1 \text{ call}) = 1 - P(0 \text{ calls}) = 1 - e^{-\lambda t} = 1 - e^{-1.25} \approx 0.713$

(d) Expected total service time $= EX \cdot 8 = 40 \text{min}$

(e) $P(X \geq 16) = 1 - \sum_{k=0}^{15} \frac{5^k}{k!} e^{-5} \approx 0.000069$

Problem 5. During a workday, an engineer monitors two independent message streams:

- Emails arrive according to a Poisson process at rate 3 per hour. Let X be the number of emails received in one hour.
- System alerts arrive independently at rate 2 per hour. Let Y be the number of alerts in one hour.

Assume $X \sim \text{Poisson}(3)$, $Y \sim \text{Poisson}(2)$, and that X and Y are independent.

(a) Find the distribution of $Z = X + Y$, the total number of messages per hour.
(b) Compute $P(Z = 4)$ and $P(Z \geq 6)$.
(c) Find $\mathbb{E}[Z]$ and $\text{Var}(Z)$.
(d) Over a two–hour period, what is the distribution of the total number of messages? (give a complete answer, incl. the parameter(s) of the distribution).
(e) The setting is now changed: In each minute, the probability that any message (email or alert) arrives is $p = 1/12$, independently of the other time intervals. Let $W \sim \text{Binom}(n = 120, p = 1/12)$ be the number of messages in two hours. Approximate $P(W \geq 6)$ using an appropriate Poisson distribution and compare the exact and approximate answers.

(a) The sum of 2 independent Poisson RVs is Poisson$(\lambda_1 + \lambda_2)$

$\therefore \; Z \sim \text{Poisson}(5)$

(b) $P(Z=4) = \frac{5^4}{24} e^{-5} = \frac{625}{24} e^{-5} \approx 0.175$

$P(Z \geqslant 6) = 1 - P(Z \leq 5) = 1 - \sum_{k=0}^{5} \frac{5^k}{k!} e^{-5} \cong 0.384$

(c) $EZ = 5$; $\text{Var}(Z) = 5$

(d) Total # of messages in 2 hrs $\sim$ Poisson$(2 \cdot 5)$ = Poisson(10)

$X \sim \text{Poisson}(6)$; $Y \sim \text{Poisson}(4)$

(e) $W \sim \text{Binom}(120, \frac{1}{12})$

as in Pt.(b)

$P(W \geqslant 6) \approx P(\text{Poisson}(10) \geqslant 6) \approx 0.933$

The exact answer $P(W \geqslant 6) = 1 - \sum_{k=0}^{5} \binom{120}{k} \left(\frac{1}{12}\right)^k \left(\frac{11}{12}\right)^{120-k} \approx 0.941$

Error ≈ 0.008

Problem 6. A box contains an unlimited supply of light bulbs. Each bulb tested is defective with probability p, and working with probability $q = 1 - p$, independently of others. Let X be the number of defective bulbs found before the first working one, not counting the working one.

(a) Write the pmf of X.

(b) Compute $\mathbb{E}[X]$ using the tail–sum formula

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} P(X \geq k).$$

(c) Suppose the tester stops after checking at most 4 bulbs (this changes the probability space!). Find the probability that at least one working bulb is found before stopping, and find the expected number of defective bulbs actually observed in this truncated experiment.

(a) $p_X(k) = p^k q, \quad k = 0, 1, 2, \ldots$

(b) $EX = \sum_{k=1}^{\infty} P(X \geqslant k) = \sum_{k=1}^{\infty} p^k = \frac{p}{q}$

(c) $P(\text{1 out of 4 working}) = 1 - P(\text{4 faulty}) = 1 - p^4$

$E(\text{def} \mid \text{observed } 4) = E(Z)$, where $Z = \min(X, 4)$

$$p_Z(k) = \begin{cases} q p^k, & k = 0, 1, 2, 3 \\ p^4, & k = 4 \end{cases}$$

Check normalization: $q + qp + qp^2 + qp^3 + p^4 = q \frac{1-p^4}{1-p} + p^4 = 1.$

$EZ = qp + 2qp^2 + 3qp^3 + 4p^4 = qp(1 + 2p + 3p^2) + 4p^4$