

Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. There is a group of n people.

(a) Fix a number k , $1 \leq k \leq n$. Find the expected number of days in a 365-day calendar year such that on each of these days exactly k people out of the group have their birthday.

(b) Find the expected number of days in a 365-day calendar year such that on each of them, at least 2 people out of this group have their birthday.

(a) Probability that k people have their birthday on a fixed day

$$= \left(\frac{1}{365}\right)^{k-1} \left(\frac{364}{365}\right)^{n-k}$$

There are $\binom{n}{k}$ ways to select such group of people.

$$\text{Ans: } \binom{n}{k} \left(\frac{1}{365}\right)^k \left(\frac{364}{365}\right)^{n-k}$$

(b) Let $p = \text{Prob}(\geq 2 \text{ people have their birthday on the same day})$

$$p = 1 - \left(\frac{364}{365}\right)^n - \frac{n}{365} \left(\frac{364}{365}\right)^{n-1} = 1 - \left(\frac{364}{365}\right)^{n-1} \left(\frac{364 - n}{365}\right)$$
$$= \mathbb{E}(\text{indicator } \mathbb{I}(\leq 1 \text{ birthdays on a given day}))$$

Use the fundamental bridge to conclude that

$$\mathbb{E}(\# \text{ of days}) = 365p.$$

Problem 2. Let X be a discrete random variable with values in $\{0, 1, 2, \dots\}$ and finite variance. Define

$$g(X) = (-1)^X.$$

(1) Show that

$$\text{Var}(g(X)) = 1 - (\mathbb{E}[(-1)^X])^2.$$

(2) Suppose $X \sim \text{FS}(p)$ (First Success) with

$$P(X = k) = p(1-p)^{k-1}, \quad k = 1, 2, \dots$$

Compute $\mathbb{E}[(-1)^X]$.

(3) Deduce a closed-form expression for $\text{Var}((-1)^X)$.

(4) Evaluate $\text{Var}((-1)^X)$ in the cases $p \rightarrow 0$ and $p \rightarrow 1$, and briefly interpret the result.

(1) $g(X)$ takes values ± 1 , so $E X = \sum_{k=0}^{\infty} (-1)^k p_X(k)$

$$E(g(X)^2) = \sum (g(X))^2 p_X(k) = 1$$

$$\text{Var}(g(X)) = E(g(X))^2 - (E X)^2 = 1 - (E X)^2$$

(2) For $X \sim \text{FS}(p)$, let $Z = g(X)$.

$$E(Z) = \sum_{k=1}^{\infty} (-1)^k p(1-p)^{k-1} = (-p) \sum_{k=1}^{\infty} (-1-p)^{k-1} = \frac{-p}{2-p}$$

(3) $\text{Var}(Z) = 1 - \left(\frac{p}{2-p}\right)^2$

(4) If $p \rightarrow 0$, then $E Z \nearrow 0$, $\text{Var}(Z) \rightarrow 1$. The partial sums

$$\sum_{k=1}^t (-1)^k p(1-p)^{k-1}, \quad t = 1, 2, 3, \dots$$

oscillate between near $(-p)$ and near 0 , approaching $-\frac{p}{2-p}$.

(5) If $p \rightarrow 1$, then $E Z \searrow -1$, $\text{Var}(Z) \rightarrow 0$. With high probability

$X=1$, i.e., $Z=-1$, approaching a constant.

Problem 3. Let $X \sim \text{Poisson}(\lambda)$.

(1) Show that

$$\mathbb{E}[X(X-1)] = \lambda^2.$$

(2) Use this to compute $\text{Var}(X)$.

(3) Let $Y = \mathbf{1}_{\{X \text{ is even}\}}$ be the indicator RV of the event $\{X \text{ is even}\}$. Compute $\mathbb{E}[Y]$.

(4) (Thinning) Suppose that each event counted by X is independently kept with probability p , and discarded otherwise. Let Z be the number of kept events. Show that $Z \sim \text{Poisson}(p\lambda)$.

$$(1) P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k=0,1,\dots$$

$$\mathbb{E}(X(X-1)) = \mathbb{E}X^2 - \mathbb{E}X = \lambda(1+\lambda) - \lambda = \lambda^2$$

[as computed in Lecture 14]

$$(2) \text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = \mathbb{E}(X(X-1)) + \mathbb{E}X - (\mathbb{E}X)^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$(3) \mathbb{E}Y = \sum_{k=0}^{\infty} 1 \cdot \frac{\lambda^k}{(2k)!} e^{-\lambda} \quad (\text{summing over all even integers } k=0,2,4,\dots)$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!}$$

Recall $e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots$; $e^{-\lambda} = 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \dots$, so

$$e^{\lambda} + e^{-\lambda} = 2 \left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots \right) = 2 \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!}$$

$$\therefore \mathbb{E}Y = e^{-\lambda} \frac{e^{\lambda} + e^{-\lambda}}{2} = \frac{1 + e^{-2\lambda}}{2}$$

(4) let Z be the # of events retained after the thinning.

$$P(Z=k) = P(Z=k | X \geq k) = \sum_{m=k}^{\infty} P(Z=k | X=m)$$

$$= \sum_{m=k}^{\infty} \binom{m}{k} p^k (1-p)^{m-k} \cdot P(X=m) = \frac{p^k}{(1-p)^k} \sum_{m=k}^{\infty} \binom{m}{k} (1-p)^m \frac{\lambda^m}{m!} e^{-\lambda}$$

$$= \frac{p^k}{(1-p)^k} e^{-\lambda} \sum_{m \geq k} \frac{((1-p)\lambda)^m}{k!(m-k)!} = \frac{p^k}{(1-p)^k} \frac{e^{-\lambda}}{k!} \sum_{m \geq k} \frac{((1-p)\lambda)^m}{(m-k)!}$$

$$= \frac{((1-p)\lambda)^k p^k}{(1-p)^k} \frac{e^{-\lambda}}{k!} \sum_{m=0}^{\infty} \frac{((1-p)\lambda)^m}{m!} = (\lambda p)^k \frac{e^{-\lambda}}{k!} \cdot e^{(1-p)\lambda} = \frac{(\lambda p)^k}{k!} e^{-\lambda p}$$

$\sim \text{Poi}(\lambda p)$

Problem 4. A sequence of independent Bernoulli random variables X_1, X_2, \dots, X_n is given, where

$$\mathbb{P}(X_i = 1) = \frac{1}{n}, \quad \mathbb{P}(X_i = 0) = 1 - \frac{1}{n}.$$

Let

$$S_n = X_1 + X_2 + \dots + X_n.$$

- (1) Compute $\mathbb{E}[S_n]$.
- (2) Compute $\text{Var}(S_n)$.
- (3) Show that for each fixed k ,

$$\mathbb{P}(S_n = k) = \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k}.$$

- (4) Show that for each fixed k ,

$$\mathbb{P}(S_n = k) \rightarrow \frac{e^{-1}}{k!} \quad \text{as } n \rightarrow \infty.$$

- (5) Conclude that for large n , S_n is approximately Poisson(1), and estimate $\mathbb{P}(S_n = 0)$.

$$(1) \quad \mathbb{E}S_n = n \mathbb{E}X_1 = n \cdot \frac{1}{n} = 1$$

$$(2) \quad \text{Start with } \mathbb{E}X_1^2 = 1^2 \cdot \frac{1}{n} + 0^2 \left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

$$\mathbb{E}S_n^2 = \mathbb{E}\left(\sum_{i=1}^n X_i\right)^2 = \sum_{i=1}^n \mathbb{E}X_i^2 + 2 \sum_{i < j} \mathbb{E}(X_i X_j)$$

We have $\mathbb{P}(X_i X_j = 1) = \frac{1}{n^2}$ and $\mathbb{P}(X_i X_j = 0) = 1 - \frac{1}{n^2}$, so $\mathbb{E}(X_i X_j) = \frac{1}{n^2}$

$$\mathbb{E}S_n^2 = 1 + 2 \binom{n}{2} \cdot \frac{1}{n^2} = 1 + \frac{n(n-1)}{n^2} = 1 + \frac{n-1}{n} = \frac{2n-1}{n}$$

$$\text{Var}(S_n) = \frac{2n-1}{n} - 1 = \frac{n-1}{n}$$

(3) Since $S_n \sim \text{Binom}(n, 1/n)$ (the number of 1's in a sequence of independent Bernoulli RVs), we have $\mathbb{P}(S_n = k) = \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k}$

(4) We know from L.14 that if $\lambda = \frac{p}{n}$ is constant as $n \rightarrow \infty$, then the binomial distribution \rightarrow Poisson(λ). In our case $\lambda = 1$, which implies the claim

$$(5) \quad \therefore \mathbb{P}(S_n = 0) \approx \frac{1}{e} \approx 0.368$$

Problem 5. Let $X \sim \text{Uniform}(0, 1)$.

- (1) Compute $\mathbb{E}[X]$ and $\text{Var}(X)$.
- (2) Let $Y = -\ln X$. Find the probability density function of Y .
- (3) Compute $\mathbb{E}[Y]$.

$$(1) \quad \mathbb{E}X = \int_0^1 x dx = \frac{1}{2}; \quad \mathbb{E}X^2 = \int_0^1 x^2 dx = \frac{1}{3}; \quad \text{Var}(X) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$(2) \quad F_Y(t) = P(-\ln X \leq t) = P(X \geq e^{-t}) = 1 - e^{-t}, \quad 0 \leq t < \infty$$

$$f_Y(t) = (F_Y(t))'_t = e^{-t}, \quad 0 \leq t < \infty$$

$$(3) \quad \mathbb{E}Y = \int_0^{\infty} t e^{-t} dt = -t e^{-t} \Big|_0^{\infty} + \int_0^{\infty} e^{-t} dt = 1$$

Problem 6. Let X_1, X_2 be independent random variables, each uniformly distributed on $(0, 1)$. Let

$$M = \max(X_1, X_2), \quad m = \min(X_1, X_2).$$

- (1) Find the probability density functions of M and m .
- (2) Compute $\mathbb{E}[M]$.
- (3) Compute $\mathbb{E}[m]$.
- (4) Compute $\mathbb{P}(X_1 + X_2 \leq 1)$.

$$(1) \quad F_M(x) = P(X_1 \leq x, X_2 \leq x) = x^2, \quad 0 \leq x \leq 1$$

$$F_m(x) = P(\min(X_1, X_2) \leq x) = 1 - P(X_1 > x, X_2 > x) = 1 - (1-x)^2 = 2x - x^2 \quad 0 \leq x \leq 1$$

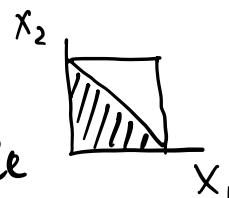
$$f_M(x) = 2x \quad \text{if } 0 \leq x \leq 1 \quad \text{and } 0 \text{ o/w}$$

$$f_m(x) = 2(1-x) \quad \text{if } 0 \leq x \leq 1 \quad \text{and } 0 \text{ o/w}$$

$$(2) \quad \mathbb{E}M = \int_0^1 2x^2 dx = \frac{2}{3}$$

$$(3) \quad \mathbb{E}m = \int_0^1 2x(1-x) dx = 1 - \frac{2}{3} = \frac{1}{3}$$

(4) $P(X_1 + X_2 \leq 1) = \frac{1}{2}$ b/c. it is the area of the hashed triangle



or:

$$P(X_1 + X_2 \leq 1) = \int_0^1 P(X_1 \leq 1-x) f_{X_2}(x) dx = \int_0^1 x dx = \frac{1}{2}$$