ENEE324, Home assignment 5. Date due April 5, 2026, 11:59pm EDT.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. There is a group of n people.

(a) Find the expected number of days in a 365-day calendar year such that on each of these days exactly k people out of the group have their birthday.

(b) Find the expected number of days in a 365-day calendar year such that on each of them, at least 2 people out of this group have their birthday.

Problem 2. Let X be a discrete random variable with values in $\{0, 1, 2, \dots\}$ and finite variance. Define

$$g(X) = (-1)^X.$$

(1) Show that

$$\mathrm{Var}(g(X)) = 1 - \left(\mathbb{E}[(-1)^X]\right)^2.$$

(2) Suppose $X \sim \mathrm{FS}(p)$ (First Success) with

$$P(X = k) = p(1-p)^{k-1}, \qquad k = 1, 2, \dots.$$

Compute $\mathbb{E}[(-1)^X]$.

(3) Deduce a closed-form expression for $\mathrm{Var}((-1)^X)$.

(4) Evaluate $\mathrm{Var}((-1)^X)$ in the cases $p \to 0$ and $p \to 1$, and briefly interpret the result.

Problem 3. Let $X \sim \mathrm{Poisson}(\lambda)$.

(1) Show that

$$\mathbb{E}[X(X-1)] = \lambda^2.$$

(2) Use this to compute $\mathrm{Var}(X)$.

(3) Let $Y = \mathbf{1}_{\{X \text{ is even}\}}$ be the indicator RV of the event $\{X \text{ is even}\}$. Compute $\mathbb{E}[Y]$.

(4) (Thinning) Suppose that each event counted by X is independently kept with probability p, and discarded otherwise. Let Z be the number of kept events. Show that $Z \sim \mathrm{Poisson}(p\lambda)$.

Problem 4. A sequence of independent Bernoulli random variables $X_1, X_2, \dots, X_n$ is given, where

$$\mathbb{P}(X_i = 1) = \frac{1}{n}, \qquad \mathbb{P}(X_i = 0) = 1 - \frac{1}{n}.$$

Let

$$S_n = X_1 + X_2 + \cdots + X_n.$$

(1) Compute $\mathbb{E}[S_n]$.

(2) Compute $\mathrm{Var}(S_n)$.

(3) Show that for each fixed k,

$$\mathbb{P}(S_n = k) = \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k}.$$

(4) Show that for each fixed k,

$$\mathbb{P}(S_n = k) \longrightarrow \frac{e^{-1}}{k!} \quad \text{as } n \to \infty.$$

(5) Conclude that for large n, S_n is approximately $\mathrm{Poisson}(1)$, and estimate $\mathbb{P}(S_n = 0)$.

Problem 5. Let $X \sim \mathrm{Uniform}(0, 1)$.

(1) Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.
(2) Let $Y = -\ln X$. Find the probability density function of Y .
(3) Compute $\mathbb{E}[Y]$.

Problem 6. Let X_1, X_2 be independent random variables, each uniformly distributed on $(0, 1)$. Let

$$M = \max(X_1, X_2), \qquad m = \min(X_1, X_2).$$

(1) Find the probability density functions of M and m.
(2) Compute $\mathbb{E}[M]$.
(3) Compute $\mathbb{E}[m]$.
(4) Compute $\mathbb{P}(X_1 + X_2 \leq 1)$.