ENEE324, Home assignment 6. Date due November 11, 2025, 11:59pm EST.

Instructor: Alexander Barg

Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. The number of particles hitting a sensing screen per second is a Poisson RV with $\lambda = 3.2$.

1. What is the probability that within a second, no particles have been registered?
2. What is the probability that there was at least one particle within a second?
3. Find the probability that there were ≥ 2 and ≤ 5 particles within a second?
4. What is the variance of the number of particles arriving at the screen?

Let $X \sim \text{Poisson}(3.2)$

1. $P(X=0) = e^{-3.2} =$

2. $P(X \geq 1) = 1 - P(X=0) = 1 - e^{-3.2}$

3. $P(2 \leq X \leq 5) = \left(\frac{(3.2)^2}{2} + \frac{(3.2)^3}{6} + \frac{(3.2)^4}{24}\right) e^{-3.2}$

4. $Var(X) = \lambda = 3.2$

Problem 2. You have a well-shuffled standard deck of 52 cards. You turn the cards face up one by one, without replacement. Recall that the deck contains 4 aces.

1. Let X_0 be the random number of cards opened before the first ace is opened. Find EX_0.
2. Let X_1 be the random number of cards between the first ace and the second ace. Find EX_1.

We claim that the expected length of each of the 2 gaps, EX_0 and EX_1 is

$$\frac{52-4}{5} = \frac{48}{5}$$

Here 5 = # gaps between the 4 aces, counting the segment of the deck before the 1st ace and after the last ace.

$$\underbrace{\quad}_{X_0} A \underbrace{\quad}_{X_1} A \underbrace{\quad}_{X_2} A \underbrace{\quad}_{X_3} A \underbrace{\quad}_{X_4}$$

This is because, by symmetry,

$$EX_0 = EX_1 = EX_2 = EX_3 = EX_4$$

More formally, let $C_1, C_2, \dots, C_{48}$ be the non-ace cards and let $I_{k,i} = \mathbb{1}(\text{card } C_k \in X_i)$, $1 \leq k \leq 48$, $0 \leq i \leq 4$

Then

$$X_i = \sum_{k=1}^{48} I_{k,i}, \quad 0 \leq i \leq 4$$

Now fix the card C_k, for some k, then since the positions of the other cards are equiprobable, $P(I_{k,i}=1)$ does not depend on i. Thus, $P(I_{k,i}=1) = \frac{1}{5}$ for all $i = 0,1,2,3,4$. Then

$$EX_i = \sum_{k=1}^{48} E I_{k,i} = \frac{48}{5} \text{ for all } i.$$

Problem 3. Use the Poisson approximation to answer the following questions related to the birthday problem.

(a) How many people are needed to have a 50% chance that at least one of them has their birthday on March 1st?

(b) How many people are needed to have a 50% chance that at least one pair of them were not only born on the same day of the year, but also were born within the same hour (out of 24 hours)? As always with birthday problems, we assume that all the days are equiprobable, and all the hours within the same day are equiprobable.

(c) With 100 people, there is a 64% chance that there is at least one set of 3 people with the same birthday. Provide a Poisson approximation for this value by considering an indicator random variable for each triplet of people (thus, 161,700 random variables). Provide another Poisson approximation by considering an indicator random variable for each of the 365 days of the year.

(a) Let $A_i = \{i^{th}$ person was born on March $1^{st}\}$; A_i for different i are independent

$P(A_i) = \frac{1}{365}$, $i = 1, 2, \ldots$

We use the **Poisson paradigm**: # of A_i occurred in a sample of n is $\approx$ Poisson(λ_n), where $\lambda_n = n/365$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P(\#A_i \text{ occurred} \geq 1) = P(\text{Poisson}(\lambda_n) \geq 1) = 1 - P(\text{Poisson}(\lambda_n) = 0)$$

So we need to find the smallest n s.t.

$$1 - e^{-\lambda_n} \geq \frac{1}{2},$$

or $\lambda_n = \ln 2$, solving for n: $n = \lceil 365 \ln 2 \rceil = 253$

(b) Let $I_{kj} = \mathbb{I}(2 \text{ persons born on hour } j \text{ of day } k)$; $P(I_{kj} = 1) = \frac{1}{365 \cdot 24} = \frac{1}{8760}$

There are $\binom{n}{2}$ pairs; prob. each pair shares (day, hour) $= \frac{1}{8760}$

Using the Poisson paradigm, $\lambda_n = \binom{n}{2} \frac{1}{8760} = \frac{n(n-1)}{2 \cdot 8760}$

As above, if $A_i = \{i^{th}$ pair shares $(D, H)\}$, then

$$P(\#A_i \text{ occurred} \geq 1) = P(\text{Poisson}(\lambda_n) \geq 1) = 1 - P(\text{Poisson}(\lambda_n) = 0)$$

$$= 1 - e^{-\lambda_n} \overset{?}{\geq} 1/2$$

This yields $\lambda_n = \ln 2$ or $n^2 - n = (\ln 2)\, 17520 \approx 12143.9$

Solving the quadratic, $n = \left\lceil \frac{1}{2} + \sqrt{\frac{1}{4} + 12143.9} \right\rceil = \lceil 110.7 \rceil = 111$

(c) Part 1: # of triplets $T = \binom{100}{3} = 161700$

Let p = Prob. that a triple shares a birthday $= \frac{1}{365^2}$

Poisson $\lambda = T \cdot p \approx 1.214$

$P(\geq 1 \text{ triple shares a b/day}) = 1 - e^{-\lambda} = 1 - e^{-1.214} \approx 0.703$

Part 2: Let $X \sim \text{Binom}(n, p)$ be the RV that counts # b/days on a particular day, where $n = 100$, $p = \frac{1}{365}$

Find

$$P(X \geq 3) = 1 - q^n - npq^{n-1} - \binom{n}{2} p^2 q^{n-2} \approx 0{,}00272$$

Or we can use a Poisson approx. to $\text{Binom}(n,p)$ with $\lambda = np = 100/365$

$$P(X \geq 3) \approx P(\text{Poisson}(\lambda) \geq 3) = 1 - P(\text{Poisson}(\lambda) \leq 2)$$

$$= 1 - e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right) \approx 0.0028$$

Either way, let us set the stage to use the Poisson paradigm:

Let $A_1 = \{\geq 3 \text{ birthdays on Jan } 1\}, \ldots, A_{365} = \{\geq 3 \text{ b/days on Dec. } 31\}$

days with ≥ 3 birthdays $= \# i$ s.t. A_i occurs

$$\sim \text{Poisson}(365 \cdot 0.0028)$$

$P(\text{at least one } A_i \text{ occurs}) = 1 - e^{-365 \cdot 0.0028} \approx 0.6401$

a very accurate approximation!

Problem 4. A point is chosen uniformly on the segment of the real line $[0, 3]$. Denote its position by X.

1. Find $E(X)$ and Var(X).
2. What is the probability that X is closer to the right end of the segment than to the left end?
3. Define the random variable $Y = X^2$. Find the pdf of Y and verify that it integrates to 1.
4. Compute EY.

1. $EX = 1.5$; $Var(X) = \frac{(3-0)^2}{12} = \frac{3}{4} = 0.75$

2. By symmetry, the answer is $\frac{1}{2}$; or $P(1.5 \leq X \leq 3) = \frac{1}{2}$

3. $P(Y \leq y) = P(X^2 \leq y) = P(0 \leq X \leq \sqrt{y}) = \frac{\sqrt{y}}{3}$, $0 \leq y \leq 9$

$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{1}{6\sqrt{y}} & 0 < y < 9 \\ 0 & y \geq 9 \end{cases}$$

$\int_0^y f_Y(t)\,dt = P(Y \leq y)$ and $P(9) = 1$ as desired

4. Use the tail property (note that $Y \geq 0$):

$$EY = \int_0^\infty P(Y > y)\,dy = \int_0^9 \left(1 - \frac{\sqrt{y}}{3}\right) dy = 9 - \frac{1}{3}\cdot\frac{2}{3}\, y^{3/2}\Big|_0^9 = 9 - 6 = 3.$$

Problem 5. Let $X \sim \mathcal{N}(5, 4)$.

1. Find $P(X > 7)$.

2. Find $P(3 < X < 7)$.

3. Which value of x satisfies $P(X \le x) = 0.975$?

4. For $Z \sim \mathcal{N}(0, 1)$ define $Y = 3Z + 2$. Find the distribution law of Y: name the law and give its mean and variance.

1. $P(X>7)$ can be found either from CDF $F_X(x) = \frac{1}{2\sqrt{2\pi}} \int_7^\infty e^{-\frac{(x-5)^2}{4}} dx$

or by transforming X to standard normal

$$Z = \frac{X - EX}{\sigma_X} = \frac{X-5}{2}; \quad X = 2Z+5$$

Then $P(X>7) = P(2Z+5>7) = P(Z>1) = 1 - \Phi(1)$

where $\Phi(x)$ = standard normal CDF.

Either way, $P(X>7) \cong 0.1587$

2. $P(3<X<7) = \int_3^7 F_X(x)dx \approx 0.6827$

3. Find x with $P(X \le x) = F_X(x) = 0.975$

$$x = F_X^{-1}(0.975)$$

or $P(2Z+5 \le x) = P\left(Z \le \frac{x-5}{2}\right) = \Phi\left(\frac{x-5}{2}\right) = 0.975$

$$x = 5 + 2\Phi^{-1}(0.975) \approx 8.92$$

4. $Y = 3Z+2 \iff Z = \frac{Y-2}{3}$

$$\left.\begin{array}{l} EY = E(3Z+2) = 0+2 = 2 \\ Var(Y) = Var(3Z+2) = 9 \end{array}\right\} \Rightarrow Y \sim \mathcal{N}(2, 9)$$

Problem 6. The 66-95-99% rule gives approximate probabilities of a normal random variable being within 1, 2, and 3 standard deviations of its mean. Derive analogous rules for the following distributions.

(a) Unif$(0, 1)$.

(b) Exp(1).

(c) Exp(1/2). Is there a single rule of this kind that applies to all exponential distributions irrespective of the parameter λ just as the 66-95-99% applies to all normal distributions irrespective of their mean μ and variance σ^2?

(a) Let $U \sim \text{Unif}(0,1)$

$$EU = \frac{1}{2}\ ;\ Var(U) = \frac{1}{12}\ ;\ \sigma_U = \frac{1}{2\sqrt{3}} \approx 0.289$$

$$P\left(\left|U - \frac{1}{2}\right| \le \sigma_U\right) = P\left(\frac{1}{2} - 0.289 \le U \le \frac{1}{2} + 0.289\right) = 0.577$$

$$P\left(\left|U - \frac{1}{2}\right| \le 2\sigma_U\right) = P\left(\frac{1}{2} - 0.578 \le U \le \frac{1}{2} + 0.578\right)$$

This covers the entire range of U, so Prob $= 1$

same for $P\left(\left|U - \frac{1}{2}\right| \le 3\sigma_U\right) = 1$

(b) We have: $X \sim \text{Exp}(1)$; $EX = 1$; $Var(X) = 1 \Rightarrow \sigma_X = 1$

$$P(0 \le X \le 2) = 1 - e^{-2} \cong 0.865$$
$$P(0 \le X \le 3) = 1 - e^{-3} \cong 0.950$$
$$P(0 \le X \le 4) = 1 - e^{-4} \cong 0.982$$

(c) For $X \sim \text{Exp}(\lambda)$, $EX = \frac{1}{\lambda}$, $Var(X) = \frac{1}{\lambda^2}$, $\sigma_X = \frac{1}{\lambda}$

$$P(|X - EX| \le m\sigma_X) = P\left(\frac{1}{\lambda} - \frac{m}{\lambda} \le X \le \frac{m+1}{\lambda}\right) =$$

$m = 1, 2$ or 3

$\frac{1}{\lambda} - \frac{m}{\lambda} \le 0$ for all $m \geq 1$, but $X \geq 0$

$$= P\left(0 \le X \le \frac{m+1}{\lambda}\right)$$

$$= CDF_X\left(\frac{m+1}{\lambda}\right) = 1 - e^{-\left(\frac{m+1}{\lambda}\right)\lambda} = 1 - e^{-(m+1)}$$

$1 - e^{-\lambda x}$

$\therefore$ For <u>any</u> λ, including $\lambda = \frac{1}{2}$, the rule is as in Part (b)