

ENEE324, Home assignment 6. Date due April 12, 2026, 11:59pm EDT.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1 (Extending Prob.6 of HW5). Let X_1, \dots, X_n be independent random variables uniformly distributed on $(0, 1)$. Let

$$M = \max(X_1, \dots, X_n).$$

- (1) Find the distribution function (cdf) and density (pdf) of M .
- (2) Compute $\mathbb{E}[M]$.
- (3) Let $G = 1 - M$ (the gap between 1 and the maximum). Compute $\mathbb{E}[G]$.
- (4) In other words, $\mathbb{E}[G]$ is the expected distance from 1 to the closest point among n random points in $(0, 1)$. How does $E(G)$ behave as $n \rightarrow \infty$?

Problem 2. Let $X \sim \mathcal{N}(0, 1)$ be the standard normal random variable.

- (1) Show that for any differentiable function g such that the expectations below exist,

$$\mathbb{E}[Xg(X)] = \mathbb{E}[g'(X)].$$

(Hint: Use integration by parts with the standard normal density.)

- (2) Use part (1) with $g(x) = x$ to compute $\mathbb{E}[X^2]$.
- (3) Use part (1) with $g(x) = x^2$ to compute $\mathbb{E}[X^3]$.
- (4) Use part (1) with $g(x) = e^{tx}$ to compute $\mathbb{E}[e^{tX}]$ for $t \in \mathbb{R}$.

Problem 3. Let X represent the score of a student on a standardized test. The distribution of scores is well approximated by a normal distribution with mean 100 and standard deviation 10.

- (1) Approximate the probability that a randomly chosen student scores above 120.
- (2) Find the cutoff score c such that approximately the top 10% of students score above c .
- (3) Suppose 5 students are selected independently. What is the probability that at least one of them scores above 120?
- (4) Let S be the minimum score among the 3 students. Compute $\mathbb{P}(S \leq 80)$.
- (5) Approximate the expected number of students (out of 1000) who score above 115.

Remark: While actual test scores are bounded, the normal distribution provides an accurate approximation for these probabilities.

Problem 4. A train departs from a station every 20 minutes starting at 6:00 a.m. A passenger arrives at the station at a random time uniformly distributed between 7:10 a.m. and 8:30 a.m.

- (1) What is the probability that the passenger waits at most 5 minutes for the next train?
- (2) What is the probability that the passenger waits at least 10 minutes?
- (3) Let W denote the waiting time (in minutes). What is the distribution of the random variable W ?
- (4) Compute the expected waiting time $\mathbb{E}[W]$.

Problem 5. Let $X \sim \text{Exp}(\lambda)$ (exponential RV with parameter λ) represent the waiting time in hours until a system experiences a failure.

Suppose that the system is observed only at integer times, and define

$$Y = \lfloor X \rfloor,$$

the integer part (floor) of X .

(1) Show that Y takes values in $\{0, 1, 2, \dots\}$ and compute

$$\mathbb{P}(Y = k), \quad k = 0, 1, 2, \dots$$

(2) Show that Y has a geometric-type distribution and identify its parameter in terms of λ .

(3) Compute $\mathbb{E}[Y]$.

(4) Compute $\mathbb{P}(X - Y > t)$ for $0 \leq t < 1$ and interpret this quantity.

Problem 6. A system consists of n identical components operating independently. Each component fails after an exponential time with rate λ , i.e., its lifetime is $\text{Exp}(\lambda)$. All components are installed at time 0.

Let T be the time at which the k -th failure occurs, where $1 \leq k \leq n$.

(1) For a fixed time $t > 0$, let $N(t)$ denote the number of components that have failed by time t . Show that $N(t) \sim \text{Binomial}(n, 1 - e^{-\lambda t})$.

(2) Express the event $\{T \leq t\}$ in terms of $N(t)$.

(3) Use part (1) to compute $\mathbb{P}(T \leq t)$.

(4) Deduce the distribution function of T .

(5) Compute $\mathbb{E}[N(t)]$ and interpret it in terms of the expected number of failures by time t .