ENEE324, Home assignment 7. Date due December 1, 2025, 11:59pm EST.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

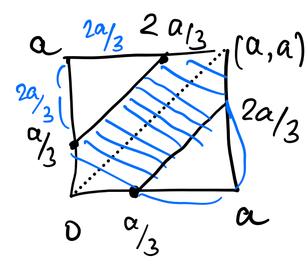
- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.

- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. Two points X_1, X_2 are chosen independently and uniformly on the line segment $[0, a]$, where $a > 0$.

(1) **Distance between two random points**
  (a) Find the probability that $|X_1 - X_2| \le \frac{a}{3}$.
  (b) Find the expected distance $E[|X_1 - X_2|]$ between the two points.
(2) **Order statistics**
  (a) Let $M = \min(X_1, X_2)$ and $N = \max(X_1, X_2)$. Find the joint pdf of (M, N).
  (b) Compute $E[M]$ and $E[N]$.
(3) **A third random point**
  (a) A third point X_3 is also chosen uniformly on $[0, a]$. Find the probability that X_3 lies between X_1 and X_2.
  (b) Given that X_3 lies between X_1 and X_2, find the expected length of the segment between the smallest and largest of the three points.

1. (a) $P\left(|X_1 - X_2| \le \frac{a}{3}\right)$



The area of the region $\overbrace{-\frac{a}{3} \le x_1 - x_2 \le \frac{a}{3}}^{A} = a^2 - \frac{4}{9}a^2 = \frac{5}{9}a^2$

$$P\left(|X_1 - X_2| \le \frac{a}{3}\right) = \iint_A \underbrace{f_{XY}(x,y)}_{=\frac{1}{a^2}} dx\,dy = \frac{5}{9}a^2 / a^2 = \frac{5}{9}$$

$(a-y)^2$

$$F_{X_1X_2}(x_1, x_2) = \frac{x_1 x_2}{a^2} \;;\; f_{X_1X_2}(x_1,x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} \frac{x_1 x_2}{a^2} = \frac{1}{a^2}, \quad 0 \le x_1, x_2 \le a$$

Let $Y = |X_1 - X_2|$; $P(Y \le y) = \iint_{|x_1 - x_2| \le y} \frac{dx\,dy}{a^2} = \frac{a^2 - (a-y)^2}{a^2} = \frac{2ay - y^2}{a^2}, \quad 0 \le y \le a$

$$f_Y(y) = 2\,\frac{a-y}{a^2}$$

$$EY = \frac{2}{a^2}\int_0^a (a-y)y\,dy = \frac{2}{a^2}\left(\frac{ay^2}{2} - \frac{y^3}{3}\right)\Big|_0^a = \frac{2}{a^2}\left(\frac{a^3}{2} - \frac{a^3}{3}\right) = \frac{a}{3}$$

2. PDF of X_1, X_2 ; $f_{X_1 X_2}(x_1, x_2) = \frac{1}{a^2}$

(a) Say $M = m, N = n, n \leq m$, then either $X_1 = n, X_2 = m$ or $X_1 = m, X_2 = n$

$\therefore f_{MN}(m, n) = \frac{2}{a^2}, \quad 0 \leq n \leq m \leq a$

(b) Marginal pdf's are found as follows

$$f_M(x) = \int_0^x f_{MN}(x, y)\,dy = \frac{2x}{a^2} ; \quad f_N(y) = \int_y^a \frac{2\,dx}{a^2} = \frac{2}{a^2}(a-y)$$

$$0 \leq x, y \leq a$$

$$\therefore EM = \int_0^a \frac{2x}{a^2}\,dx = \frac{2}{3}a ; \quad EN = \int_0^a \frac{2y(a-y)}{a^2} = \frac{a}{3}$$

3.

(a) Let $X_i = x_i$, $i = 1, 2, 3$. All 6 permutations

$$x_1 < x_2 < x_3, \ldots, x_3 < x_2 < x_1$$

are equally probable; for 2 of them x_3 is between x_1 and x_2

$\therefore P(X_3 \text{ between } X_1 \text{ and } X_2) = \frac{1}{3}$

(b) We are given that X_3 is between X_1 and X_2; then the expected length $= E|X_1 - X_2|$ and it does not depend on X_3 since X_3 is chosen independently.

As found above, $E|X_1 - X_2| = \frac{a}{3}$

Problem 2. Let X_1 and X_2 be independent random variables, each exponentially distributed with parameter $\lambda > 0$. That is, $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.

(1) Find the moment generating function (mgf) of X_1.
(2) Using independence, find the mgf of $S = X_1 + X_2$.
(3) Identify the distribution of S and its parameters.
(4) Using the mgf, find $E[S]$ and $\mathrm{Var}(S)$.

(1) $M_S(t) = \frac{\lambda}{\lambda - t}$, $t < \lambda$ found in class; see lect. notes L.20

(2) $M_S(t) = M_{X_1}(t) M_{X_2}(t) = \frac{\lambda^2}{(\lambda - t)^2}$, $t < \lambda$

(3) The law of S is called gamma distribution of order 2; $\Gamma(2, \lambda)$

(4) $M_S'(t) = \frac{2\lambda^2}{(\lambda - t)^3}$; $ES = M_S'(0) = \frac{2}{\lambda}$

$M_S''(t) = \frac{6\lambda^2}{(\lambda - t)^4}$; $ES^2 = M''(0) = \frac{6}{\lambda^2}$

$\mathrm{Var}(S) = ES^2 - (ES)^2 = \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}$

Problem 3. A stick of total length 1 is broken at a random point U, where U is uniformly distributed on $(0, 1)$. Let the break divide the stick into two pieces of lengths U and $1 - U$.

(1) Find the probability density function (pdf) of the random variable U.
(2) Let $L = \max(U, 1 - U)$ be the length of the longer piece. Find the pdf of L and compute $E[L]$.
(3) Let $S = \min(U, 1 - U)$ be the length of the shorter piece. Find $E[S]$.
(4) Fix a point $p \in (0, 1)$ along the original stick. Determine the expected length of the piece that contains the point p.
(5) For what value of p is this expected length the smallest? Interpret the result geometrically.

(1) $U \sim \mathrm{Unif}[0,1]$; $f_U(x) = 1$, $0 \le x \le 1$ and 0 o/w

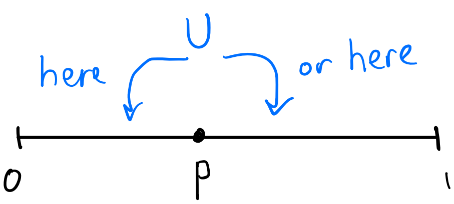
(2) $P(L \le x) = P(U \le x, \underset{U \ge 1-x}{1-U \le x}) = P(1-x \le U \le x) = 2x-1$, $x \ge 1/2$; 0 o/w

$$f_L(x) = 2\,\mathbb{1}\left(\tfrac{1}{2} \le x \le 1\right)$$

$$EL = 2\int_{1/2}^{1} x\,dx = x^2\Big|_{1/2}^{1} = \frac{3}{4}$$

(3) Since $S = 1 - L$, $ES = \frac{1}{4}$

(4) Let ℓ be the expected length



If $U \le p$, then $E\ell = E[1-U] = \int_0^p (1-x)\,dx = x - \frac{x^2}{2}\Big|_0^p = p - \frac{p^2}{2}$

If $U > p$ then $E\ell = E[U] = \int_p^1 x\,dx = \frac{1}{2} - \frac{p^2}{2}$; Ans: $E\ell = \frac{1}{2} + p - p^2$

(5) The minimum of $E\ell$ occurs for $p = 0, 1$ and equals $\frac{1}{2}$ (and the maximum for $p = \frac{1}{2}$). This happens b/c with probability $\frac{1}{2}$ the piece that contains it is the shortest. Thus for $p = 0$

$$E\ell = \frac{1}{2} \cdot EL + \frac{1}{2} ES = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4} = \frac{1}{2}$$

Problem 4. Let X be an exponential random variable with parameter $\lambda > 0$, i.e.

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

(1) Define $Y = \ln(1 + X)$. Find the probability density function (pdf) of Y.
(2) Compute $E[Y]$ in terms of λ.
(3) Define $Z = e^{-X}$. Find the pdf of Z and determine its support.
(4) Without calculation, explain whether Z has a larger or smaller mean than X.

(1) Change of variables formula will be covered later, so let us find the answer without using it:

$$P(Y \leq y) = P(\ln(1+X) \leq y) = P(X \leq e^y - 1) = F_X(e^y - 1) = 1 - e^{-\lambda(e^y-1)}$$

$$f_Y(y) = F_Y'(y) = \lambda e^{y - \lambda(e^y-1)}, \quad y \geq 0 \quad (\text{since } \ln(1+x) \geq 0 \text{ for all } x \geq 0)$$

(2) $$EY = \lambda \int_0^\infty y e^{y - \lambda(e^y - 1)} dy$$

This integral cannot be expressed in elementary functions
Can be found numerically for any $\lambda > 0$

(3) Since $X \geq 0$, e^{-X} can be anywhere between 1 and 0.

Z is supported on $(0,1)$

$$P(Z \leq z) = P(e^{-X} \leq z) = P(X \geq -\ln z) = e^{\lambda \ln z} = z^\lambda, \quad 0 \leq z \leq 1$$

$$f_Z(z) = \lambda z^{\lambda - 1}, \quad 0 \leq z \leq 1 \text{ and } 0 \text{ o/w}$$

Finding EZ is easy, but note that $e^{-x} \leq x$ for all except small values of x. Moreover, large values of X yield small values of Z. Thus, if large values of X have relatively high probability, i.e, if λ is small, EZ will be small, and in particular, smaller than $EX = \frac{1}{\lambda}$. For large λ, this relation will flip.
Checking these informal observations, compute

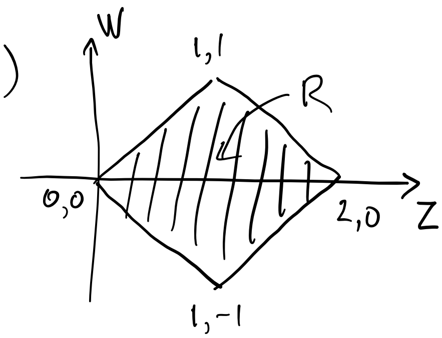
$$EZ = \lambda \int_0^1 z^\lambda \, d\lambda = \frac{\lambda}{\lambda + 1}. \text{ We find } \quad EZ \underset{\lambda > 1.618}{\overset{\lambda < 1.618}{\lesseqgtr}} EX.$$

Problem 5. Let X and Y be independent random variables, each uniformly distributed on $(0, 1)$. Define new random variables

$$Z = X + Y, \qquad W = X - Y.$$

(1) Find the range (support) of the random vector (Z, W). Sketch or describe it geometrically.
(2) Find the joint pdf $f_{Z,W}(z, w)$ and verify that Z and W are independent.
(3) Compute $\mathrm{Cov}(Z, W)$ and verify that Z and W are uncorrelated.
(4) Express X and Y in terms of Z and W, and determine whether X and Y remain independent when expressed this way.

(1)



$$R = \{0 \le z \le 2 \; ; \; \max(-z, z-2) \le w \le \min(z, 2-z)\}$$

(2) We have $f_{X,Y}(x,y) = 1$, $0 \le x \le 1, 0 \le y \le 1$

To find $f_{Z,W}(z,w)$, express X and Y via Z and W

$$X = \frac{1}{2}(Z+W) \; ; \; Y = \frac{1}{2}(Z-W)$$

Jacobian $\quad |\det J| = \left| \det\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \right| = \frac{1}{2}$

$$\therefore f_{Z,W}(z,w) = \begin{cases} |\det J| \, f_{X,Y}\left(\frac{1}{2}(Z+W), \frac{1}{2}(Z-W)\right) = \frac{1}{2} & \text{if } (z,w) \in R \\ 0, & \text{if } (z,w) \notin R \end{cases}$$

(3) Marginals

$$f_Z(z) = \begin{cases} \int_{-z}^{z} \frac{1}{2} \, dw = \frac{1}{2} w \Big|_{-z}^{z} = z, & 0 \le z \le 1 \\ \int_{z-2}^{2-z} \frac{1}{2} \, dw = \frac{1}{2} w \Big|_{z-2}^{2-z} = \frac{1}{2}(2-z-z+2) = 2-z, & 1 \le z \le 2 \end{cases}$$

$$f_W(w) = \begin{cases} 1+w, & -1 \le w \le 0 \\ 1-w, & 0 \le w \le 1 \end{cases}$$

$\therefore$ Z and W are not independent since generally

$$f_{Z,W}(z,w) \neq f_Z(z) f_W(w)$$

(4) $\mathrm{Cov}(Z,W) = E[ZW] - EZ\,EW$

$EZ = E(X+Y) = 1$; $EW = 0$

$E[ZW] = E[(X+Y)(X-Y)] = EX^2 - EY^2 = 0$

$\therefore$ $\mathrm{Cov}(Z,W) = 0$

Z, W are uncorrelated

(5) $X = \frac{1}{2}(Z+W)$; $Y = \frac{1}{2}(Z-W)$

It suffices to check that $Z+W$ and $Z-W$ are independent

We could find joint pdf of $Z+W$ and $Z-W$ directly, but reversing the relations, we claim that X and Y are the RVs we started with, and thus are independent

Problem 6. The joint probability density function of random variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 6(1-y), & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Verify that $f_{X,Y}$ is a valid joint pdf.
(2) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.
(3) Find the conditional pdf $f_{X|Y}(x|y)$.
(4) Compute $E[X|Y=y]$ and then $E[X]$.
(5) Are X and Y independent? Explain.

(1) Normalization

$$\int_0^1\int_x^1 6(1-y)\,dy\,dx = \int_0^1\left(6y-3y^2\Big|_x^1\right)dx = \int_0^1(6-6x-3+3x^2)\,dx$$

$$= 3-(3x^2-x^3)\Big|_0^1 = 1 \checkmark$$

$f_{XY}(x,y) \geq 0$ for all x,y. $\therefore$ valid joint pdf

(2) $f_X(x) = \int_x^1 6(1-y)\,dy = 3-6x+3x^2, \quad 0<x<1$

$f_Y(y) = \int_0^y 6(1-y)\,dx = 6(1-y)x\Big|_0^y = 6y(1-y), \quad 0<x<1$

(3) $f_{X|Y}(x|y) = \dfrac{6(1-y)}{6y(1-y)} = \dfrac{1}{y}, \quad 0<x<y$ and 0 o/w

(4) $E[X|Y=y] = \int_0^y \frac{x}{y}\,dx = \frac{x^2}{2y}\Big|_0^y = \frac{y}{2}$

$E[X] = E[E[X|Y]] = \int_0^1 \frac{y}{2}\cdot 6y(1-y)\,dy = \frac{1}{2}EY = \frac{1}{2}\left(6\left(\frac{1}{3}-\frac{1}{4}\right)\right) = \frac{1}{4}$

(5) Since $f_{XY}(x,y) \neq f_X(x)f_Y(y)$ for all or most (x,y)

X and Y are not independent