ENEE324, Home assignment 7. Date due December 1, 2025, 11:59pm EST.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.

- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. Two points X_1, X_2 are chosen independently and uniformly on the line segment $[0, a]$, where $a > 0$.

(1) **Distance between two random points**
  (a) Find the probability that $|X_1 - X_2| \le \frac{a}{3}$.
  (b) Find the expected distance $E[|X_1 - X_2|]$ between the two points.
(2) **Order statistics**
  (a) Let $M = \min(X_1, X_2)$ and $N = \max(X_1, X_2)$. Find the joint pdf of (M, N).
  (b) Compute $E[M]$ and $E[N]$.
(3) **A third random point**
  (a) A third point X_3 is also chosen uniformly on $[0, a]$. Find the probability that X_3 lies between X_1 and X_2.
  (b) Given that X_3 lies between X_1 and X_2, find the expected length of the segment between the smallest and largest of the three points.

Problem 2. Let X_1 and X_2 be independent random variables, each exponentially distributed with parameter $\lambda > 0$. That is, $f_X(x) = \lambda e^{-\lambda x}$ for $x \ge 0$.

(1) Find the moment generating function (mgf) of X_1.
(2) Using independence, find the mgf of $S = X_1 + X_2$.
(3) Identify the distribution of S and its parameters.
(4) Using the mgf, find $E[S]$ and $\mathrm{Var}(S)$.

Problem 3. A stick of total length 1 is broken at a random point U, where U is uniformly distributed on $(0, 1)$. Let the break divide the stick into two pieces of lengths U and $1 - U$.

(1) Find the probability density function (pdf) of the random variable U.
(2) Let $L = \max(U, 1 - U)$ be the length of the longer piece. Find the pdf of L and compute $E[L]$.
(3) Let $S = \min(U, 1 - U)$ be the length of the shorter piece. Find $E[S]$.
(4) Fix a point $p \in (0, 1)$ along the original stick. Determine the expected length of the piece that contains the point p.
(5) For what value of p is this expected length the smallest? Interpret the result geometrically.

Problem 4. Let X be an exponential random variable with parameter $\lambda > 0$, i.e.

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \ge 0.$$

(1) Define $Y = \ln(1 + X)$. Find the probability density function (pdf) of Y.
(2) Compute $E[Y]$ in terms of λ.
(3) Define $Z = e^{-X}$. Find the pdf of Z and determine its support.
(4) Without calculation, explain whether Z has a larger or smaller mean than X.

Problem 5. Let X and Y be independent random variables, each uniformly distributed on $(0, 1)$. Define new random variables

$$Z = X + Y, \qquad W = X - Y.$$

(1) Find the range (support) of the random vector (Z, W). Sketch or describe it geometrically.
(2) Find the joint pdf $f_{Z,W}(z, w)$ and verify that Z and W are independent.
(3) Compute $\mathrm{Cov}(Z, W)$ and verify that Z and W are uncorrelated.
(4) Express X and Y in terms of Z and W, and determine whether X and Y remain independent when expressed this way.

Problem 6. The joint probability density function of random variables X and Y is given by

$$f_{X,Y}(x, y) = \begin{cases} 6(1 - y), & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Verify that $f_{X,Y}$ is a valid joint pdf.
(2) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.
(3) Find the conditional pdf $f_{X|Y}(x|y)$.
(4) Compute $E[X|Y = y]$ and then $E[X]$.
(5) Are X and Y independent? Explain.