ENEE324, Home assignment 7. Date due November 22, 11:59pm EST.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Notation: $\mathrm{Exp}(\lambda)$ has density $\lambda e^{-\lambda x}$, $x \ge 0$; $\mathrm{Pois}(\lambda)$ has pmf $e^{-\lambda}\lambda^k/k!$; $\mathcal{N}(\mu, \sigma^2)$ has mean μ and variance σ^2; Φ is the standard $\mathcal{N}(0,1)$ CDF; $M_X(t) = \mathbb{E}[e^{tX}]$ is the MGF.

Problem 1. Cars arrive at an intersection as a Poisson process with rate 10 per minute. Each car, independently of everything else, turns left with probability 0.3.

(1) Find the distribution of the number of left-turning cars in a 2-minute interval, and the probability that there are none.
(2) Given that exactly 5 cars arrived during a particular minute, find the probability that at least 2 of them turned left.
(3) Let L and R be the numbers of cars in one minute that do and do not turn left. Are L and R independent? Find $\mathrm{Cov}(L, R)$ and $\mathrm{Cov}(L, L+R)$.
(4) Independently of the cars, trucks arrive at the same intersection as a Poisson process with rate 2 per minute. Find the expected time until the next vehicle (car or truck) and the probability that it is a truck.
(5) Given that exactly one car arrived during a particular minute, find the probability that it arrived during the last 15 seconds of that minute.

Problem 2. (1) Let X be uniform on the four values $\{1, 2, 3, 10\}$. Find $\mathbb{E}[X]$, $\mathrm{Var}(X)$, and all medians of X.
(2) Compute $g(c) = \mathbb{E}|X - c|$ for $c = 1, 2, 2.5, 3, 4$, and find the set of all c that minimize g. Compare the mean-square errors $\mathbb{E}(X-c)^2$ for $c = \mathbb{E}[X]$ and $c = 2.5$.
(3) Let $X_1, \dots, X_n$ be independent copies of a random variable with mean μ and variance σ^2, $\bar{X}_n = \frac{1}{n}\sum_i X_i$, and $S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$. Prove that $\mathbb{E}[S_n^2] = \sigma^2$. (Hint: first show that $\sum_i (X_i - \bar{X}_n)^2 = \sum_i X_i^2 - n\bar{X}_n^2$, and use $\mathbb{E}[\bar{X}_n^2] = \mathrm{Var}(\bar{X}_n) + \mu^2$.)
(4) Four measurements gave the values $1, 2, 3, 10$. Compute the sample mean, the sample median, and S_4^2.

Problem 3. (1) A random variable X has MGF $M_X(t) = \frac{1}{3}e^{-t} + \frac{1}{2} + \frac{1}{6}e^{2t}$. Find the pmf of X, and compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ by differentiating M_X.
(2) Let $T = X_1 + \cdots + X_n$, where $X_1, \dots, X_n$ are independent $\mathrm{Exp}(\lambda)$. Find $M_T(t)$, and use it to compute $\mathbb{E}[T]$ and $\mathrm{Var}(T)$.
(3) A random variable Y has MGF $M_Y(t) = e^{5t+8t^2}$ for all t. Identify the distribution of Y and express $P(Y > 9)$ through Φ.

Problem 4. Let $X \sim \mathrm{Pois}(\lambda)$, with MGF $M_X(t) = e^{\lambda(e^t - 1)}$.

(1) Use the Chernoff bound $P(X \ge a) \le e^{-ta} M_X(t)$, $t > 0$, to show that for $a > \lambda$

$$P(X \ge a) \le e^{-\lambda}\Big(\frac{e\lambda}{a}\Big)^a.$$

(Minimize the right-hand side over t.)

(2) For $\lambda = 10$ and $a = 20$, evaluate this bound and compare it with the Chebyshev bound on $P(|X - 10| \geq 10)$ and with the exact value $P(X \geq 20) \approx 0.0035$.

Problem 5. Two fair dice are rolled. Let X be the smaller and Y the larger of the two numbers (so $X = Y$ on a double).

(1) Find the joint pmf $p_{XY}(x, y)$ and the marginal pmf of X.
(2) Compute $P(Y - X \geq 3)$.
(3) Find the conditional pmf of Y given $X = 4$, and $\mathbb{E}[Y \mid X = 4]$.
(4) Are X and Y independent? Justify with one pair (x, y).
(5) Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $\mathrm{Cov}(X, Y)$. (Hint: $XY = D_1 D_2$, where D_1, D_2 are the two dice.) Given that $\mathrm{Var}(X) = \mathrm{Var}(Y) = \frac{2555}{1296}$, find the correlation coefficient $\rho(X, Y)$.

Problem 6. The pair (X, Y) has joint density $f(x, y) = c\,x$ for $0 < y < x < 1$, and $f(x, y) = 0$ otherwise.

(1) Find c, and the marginal densities f_X and f_Y.
(2) Are X and Y independent? Justify.
(3) Find the conditional density $f_{Y|X}(y \mid x)$ and $\mathbb{E}[Y \mid X = x]$.
(4) Compute $P(X + Y < 1)$.
(5) Compute $\mathrm{Cov}(X, Y)$ and the correlation coefficient $\rho(X, Y)$.

Problem 7. Light bulbs in a store come from batch A (lifetime Exp with mean 1000 hours) with probability 0.8, or from batch B (lifetime Exp with mean 200 hours) with probability 0.2. You buy a bulb, not knowing its batch.

(1) The bulb fails at exactly $T = 300$ hours. Find the probability that it came from batch B.
(2) Instead, suppose you only know that the bulb is still working at 300 hours. Find the probability that it came from batch B.
(3) In the situation of part (2), find the probability that the bulb works for at least 500 more hours, and compare with the probability that a new bulb works for at least 500 hours. Why do they differ, although each batch has memoryless lifetimes?