ENEE324, Home assignment 8. Date due December 10, 2025, 11:59pm EST.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- Justification of solutions is required.
- 5 problems, Each problem is worth 10 points unless noted otherwise.

Problem 1. Let X be a continuous random variable with pdf

$$f_X(x) = 2x, \quad 0 \le x \le 1.$$

Suppose a coin is tossed once, where the probability of heads depends on X: given $X = x$, the probability of heads is

$$P(\text{Heads}|X = x) = x.$$

(1) Compute the unconditional probability of heads using the continuous law of total probability:

$$P(\text{Heads}) = \int_0^1 P(\text{Heads}|X = x) f_X(x)\, dx.$$

(2) Compute the conditional pdf of X given that the coin shows heads, $f_{X|\text{Heads}}(x)$, using the continuous Bayes formula:

$$f_{X|\text{Heads}}(x) = \frac{f_X(x)P(\text{Heads}|X = x)}{P(\text{Heads})}.$$

(3) Using the conditional pdf, compute $E[X|\text{Heads}]$.

(4) Suppose instead the coin shows tails. Compute $f_{X|\text{Tails}}(x)$ and $E[X|\text{Tails}]$.

(1) $P(H) \overset{\text{LOTP}}{=} \int_0^1 P(H|X=x) f_X(x)\, dx = 2\int_0^1 x^2\, dx = \frac{2}{3}$

(2) $f_{X|H}(x) = \frac{3}{2} \cdot 2x \cdot x = 3x^2$, $0 \le x \le 1$ and 0 o/w

(3) $E[X|H] = \int_0^1 3x^3\, dx = \frac{3}{4}$

(4) $P(T|X=x) = 1-x$; $P(T) = 1 - P(H) = \frac{1}{3}$

$f_{X|T}(x) = \frac{f_X(x)\, P(T|X=x)}{P(T)} = \frac{2x\cdot(1-x)}{1/3} = 6x(1-x)$, $0 \le x \le 1$

$E[X|T] = \int_0^1 6x^2(1-x)\, dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = 6 \cdot \frac{1}{12} = \frac{1}{2}$.

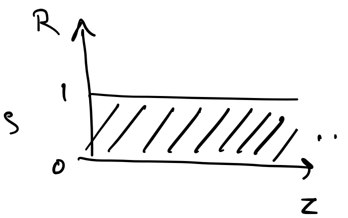
Problem 2. Two independent machines, A and B, each require a random time to complete a task. Let X and Y denote these times, both exponentially distributed with rate $\lambda > 0$.

Define the total completion time and the relative share of A in the total time by

$$Z = X + Y, \qquad R = \frac{X}{X+Y}.$$

(1) Determine the range of possible values of *the pair* (Z, R) as a region on the plane (see textbook, Sec.8.1).
(2) Derive the joint pdf $f_{Z,R}(z, r)$ using the transformation from (X, Y) to (Z, R).
(3) Find the marginal pdfs $f_Z(z)$ and $f_R(r)$, and identify the distributions of Z and R.
(4) Show that Z and R are independent, and explain intuitively why this makes sense in the context of the problem.
(5) Compute $E[Z]$, $\mathrm{Var}(Z)$, and $E[R]$.

(1) Clearly, $0 < Z < \infty$, and $0 < R < 1$, so the region is



(2) We have $\left.\begin{array}{l} z = x + y \\ rx + ry = x \end{array}\right\}$ $\begin{array}{l} x = z - y \\ r(z-y) + ry = rz = z - y \end{array}$ $\left.\right\}$ $\begin{array}{l} y = z(1-r) \\ x = z - z(1-r) = zr \end{array}$

$$|\det J| = \left|\begin{vmatrix} r & z \\ 1-r & -z \end{vmatrix}\right| = |-rz - (1-r)z| = z$$

$$f_{ZR}(z,r) = f_{XY}(x,y)\left|\frac{\partial(x,y)}{\partial(z,r)}\right| = \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y} z = \lambda^2 e^{-\lambda(zr + z(1-r))} \cdot z$$

$$= \lambda^2 z e^{-\lambda z}$$

(3) $f_Z(z) = \lambda^2 z e^{-\lambda z}$ = Gamma$(2, z)$ = sum of 2 independent exponentials

$f_R(r) = 1$ - Unif$[0,1]$; Z and R are independent.

This makes sense b/c X and Y are independent, and X is equally likely to take any share of the $X+Y$ value by the memoryless property.

(4) $$EZ = \lambda^2 \int_0^\infty \underbrace{z^2}_{u} \underbrace{e^{-\lambda z} dz}_{dv} = \underbrace{-\lambda z^2 e^{-\lambda z}\Big|_0^\infty}_{=0} + 2\lambda \int_0^\infty z e^{-\lambda z} dz = \frac{2}{\lambda}$$

$$EZ^2 = \int_0^\infty \lambda^2 z^3 e^{-\lambda z} dz \overset{\text{by parts}}{=} -\lambda z^3 e^{-\lambda z}\Big|_0^\infty + 3\lambda \int_0^\infty z^2 e^{-\lambda z} dz = \frac{6}{\lambda^2}$$ (the last $\int$ found above)

$$\mathrm{Var}(Z) = EZ^2 - (EZ)^2 = \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2} \; ; \quad ER = \frac{1}{2}$$

Problem 3. A well-shuffled standard deck of 52 cards contains 4 aces. You draw 3 cards at random without replacement and let X be the number of aces among the three cards.

(1) Compute the unconditional expectation $E[X]$.
(2) Let A be the event "at least one ace is drawn" (i.e. $A = \{X \geq 1\}$). Compute the conditional expectation $E[X|A]$.
(3) Let B be the event "the first card drawn is an ace". Compute $E[X|B]$.
(4) Compare $E[X|A]$ and $E[X|B]$. Which is larger and why? Give a short intuitive explanation (you may also support your answer with the numerical values).

(1) $X \sim \text{HGeom}(\overset{n}{3}, \overset{w}{4}, \overset{b}{48})$

$$EX = \frac{nw}{w+b} = \frac{3\cdot 4}{52} = \frac{3}{13}$$ as found in L.9.

Or, we can find the PMF of X.

$$P(X=2) = \frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}} = \frac{\cancel{6}\cdot 48\cdot 6}{52\cdot \underset{17}{\cancel{51}}\cdot \underset{25}{\cancel{50}}} = \frac{48\cdot 6}{52\cdot 17\cdot 25} = \frac{72}{5525}$$

$$P(X=1) = \frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}} = \frac{1128}{5525}$$

$$P(X=3) = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{4\cdot 6}{52\cdot 51\cdot 50} = \frac{1}{5525}$$

and compute $EX = \frac{1}{5525}(1128 + 2\cdot 72 + 3) = \frac{1275}{5525} = \frac{7}{13}$

(2) $A = \{X \geq 1\}$ $P(X\geq 1) = 1 - P(X=0) = 1 - \frac{48}{52}\cdot\frac{47}{51}\cdot\frac{46}{50} = \frac{1201}{5525}$

$$P(X=1|X\geq 1) = \frac{P(X=1 \cap X\geq 1)}{P(X\geq 1)} = \frac{P(X=1)}{P(X\geq 1)} = \frac{1128}{1201} \approx 0.938$$

$$P(X=2|X\geq 1) = \frac{72}{1201} \approx 0.059$$

$$P(X=3|X\geq 1) = \frac{1}{1201} \approx 0.0008$$

$$E(X|A) = \frac{1}{1201}(1128 + 2\cdot 72 + 3) = \frac{1275}{1201} \approx 1.0616$$

(3) $B = \{1^{st} \text{ card ace}\}$

Once one ace is removed, we are left with 51 cards of which three are aces.

$\#$ aces in the remaining 2 draws $\sim HGeom(2,3,48)$

Expected $\#$ of aces in these 2 draws $= \frac{2\cdot 3}{51} = \frac{2}{17}$

$E(X|B) = 1 + \frac{2}{17} = \frac{19}{17} \approx 1.118$

(4) We see that $E(X|B) > E(X|A)$

To explain why, let us compute the conditional PMF. The 1^{st} draw is an ace; we are left with 2 draws $\sim HGeom(2,3,48)$

If they produce 0, 1 or 2 aces, then $X = 1, 2, 3$ resp.

0

$$P(X=1|B) = \frac{\binom{48}{2}}{\binom{51}{2}} = \frac{48\cdot 47}{51\cdot 50} = \frac{8\cdot 47}{17\cdot 25} = \frac{376}{425} \approx 0.885$$

1

$$P(X=2|B) = \frac{\binom{3}{1}\binom{48}{1}}{\binom{51}{2}} = \frac{6\cdot 48}{51\cdot 50} = \frac{6\cdot 8}{17\cdot 25} = \frac{48}{425} = \frac{144}{1275} \approx 0.113$$

2

$$P(X=3|B) = \frac{1}{\binom{51}{2}} = \frac{1}{51\cdot 25} = \frac{1}{1250+25} = \frac{1}{1275} \approx 0.0007$$

We see that chances to get 2 aces are twice as high under B than under A, and this shifts the expectation.

Problem 4. Let X and Y be continuous random variables with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Verify that $f_{X,Y}(x,y)$ is a valid joint pdf.
(2) Let $g(X,Y) = X^2Y + Y^2$. Compute the expectation

$$E[g(X,Y)] = \int_0^1 \int_0^1 g(x,y) f_{X,Y}(x,y)\, dx\, dy.$$

(3) Let $h(X,Y) = X + Y$. Compute $E[h(X,Y)^2]$.
(4) Compute $\mathrm{Cov}(X,Y)$ and $\mathrm{Var}(X+Y)$ using your results from above.

(1) $\int_0^1\int_0^1 4xy\, dx\, dy = 4 \cdot \frac{x^2}{2}\Big|_0^1 \frac{y^2}{2}\Big|_0^1 = 1$

(2) $E\, g(X,Y) = \int_0^1\int_0^1 (x^2y+y^2)\, 4xy\, dx\, dy = \int_0^1\int_0^1 (4x^3y^2 + 4xy^3)\, dx\, dy$

$= \int_0^1 (x^4y^2 + 2x^2y^3)\Big|_0^1 dy = \int_0^1 (y^2 + 2y^3)\, dy = \frac{1}{3} + \frac{2}{4} = \frac{5}{6}$

(3) $E(h(X,Y))^2 = \iint (x+y)^2 \cdot 4xy\, dx\, dy = \iint (4x^3y + 8x^2y^2 + 4xy^3)\, dx\, dy$

$= \int \left(x^4y + \frac{8}{3}x^3y^2 + \frac{4}{2}x^2y^3\right)\Big|_0^1 dy = \int_0^1 \left(y + \frac{8}{3}y^2 + 2y^3\right) dy = \frac{1}{2} + \frac{8}{9} + \frac{1}{2} =$

$= 1 + \frac{8}{9} = \boxed{\frac{17}{9}}$

(4) $f_X(x) = \int_0^1 4xy\, dy = 2x$; $f_Y(y) = 2y$

$EX = \int 2x^2\, dx = \frac{2}{3} = EY$

$E(XY) = \iint xy \cdot 4xy\, dx\, dy = 4\iint x^2y^2\, dx\, dy = \frac{4}{3}\int_0^1 y^2\, dy = \frac{4}{9}$

$\mathrm{Cov}(X,Y) = E(XY) - EX\, EY = 0$

$(E(X+Y))^2 = \left(\frac{2}{3} + \frac{2}{3}\right)^2 = \frac{16}{9}$ $\quad E(X+Y)^2 = \frac{68}{36}$ by pt. (3)

$Var(X+Y) = (E(X+Y))^2 - E((X+Y)^2) = \frac{4}{36} = \frac{1}{9}$

Problem 5. Two fair dice are rolled. Let X be the outcome of the first die and Y the outcome of the second die.

(1) Let $g(X,Y) = X \cdot Y + X$. Compute the expectation
(2) Let $h(X,Y) = X + Y$. Compute $E[h(X,Y)^2]$.
(3) Compute $\mathrm{Cov}(X,Y)$ and $\mathrm{Var}(X+Y)$.
(4) Let $I = \mathbf{1}(X = Y)$ be the indicator that both dice show the same number. Compute $E[I]$ and $E[g(X,Y)|I = 1]$.

(1) $EX = EY = 3.5$ $\left(\text{In full, } \frac{1}{6}(1+2+3+4+5+6) = \frac{1}{6}\frac{6\cdot 7}{2} = \frac{21}{6} = \frac{7}{2}\right)$

$1 \le X \le 6$

$1 \le Y \le 6$

$$P(XY=k) = \begin{cases} \frac{1}{36} & \text{if } k = 1, 9, 16, 25, 36 \\ \frac{2}{36} & \text{if } k = 2, 3, 5, 8, 10, 15, 18, 20, 24, 30 \\ \frac{3}{36} & \text{if } k = 4 \\ \frac{4}{36} & \text{if } k = 6 \text{ or } 12 \end{cases}$$

$5\cdot 1$ $5.$
$10\cdot 2$ 20
$1\cdot 3$ 3
$2\cdot 4$ 8

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

$$EXY = \frac{1}{36}(1+9+16+25+36+2(2+3+5+8+10+15+18+20+24+30)$$
$$+ 3\cdot 4 + 4(6+12))$$
$$= \frac{1}{36}(87 + 2\cdot 135 + 12 + 72) = \frac{441}{36} = \frac{49}{4}$$

$\times 2$

$$\boxed{E(XY+X) = \frac{49}{4} + \frac{7}{2} = \frac{63}{4}}$$

(2) $E((X+Y))^2 = EX^2 + 2EXY + EY^2 = \frac{91}{3} + \frac{98}{4} = \frac{364+294}{12} = \frac{658}{12} = \boxed{\frac{329}{6}}$

$EX^2 = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6} = EY^2$

(3) $\mathrm{Cov}(X,Y) = 0$ since X and Y are independent

$\mathrm{Var}(X) = \mathrm{Var}(Y) = \frac{91}{6} - \frac{49}{4} = \frac{364-294}{24} = \frac{70}{24} = \frac{35}{12}$

$\mathrm{Var}(X+Y) = 2\,\mathrm{Var}(X) = \frac{35}{6}$

(4) $EI = P(X=Y) = \frac{6}{36} = \frac{1}{6}$. $P(X\cdot Y = k^2 | X=Y) = \frac{1}{6}$, $k = 1,2,3,4,5,6$

$E(XY + X | X=Y) = \frac{1}{6}(2+6+12+20+30+42) = \frac{112}{6} = \frac{56}{3}$.

$X(Y+1)$