ENEE324, Home assignment 8. Date due December 10, 2025, 11:59pm EST.

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Please upload your work as a **single PDF file** to ELMS (under the ”Assignments” tab)

• Submissions on paper or by email will not be accepted.
• Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.

• Justification of solutions is required.
• 5 problems, Each problem is worth 10 points unless noted otherwise.

Problem 1. Let X be a continuous random variable with pdf

$$f_X(x) = 2x, \quad 0 \le x \le 1.$$

Suppose a coin is tossed once, where the probability of heads depends on X: given $X = x$, the probability of heads is

$$P(\text{Heads}|X = x) = x.$$

(1) Compute the unconditional probability of heads using the continuous law of total probability:

$$P(\text{Heads}) = \int_0^1 P(\text{Heads}|X = x) f_X(x)\, dx.$$

(2) Compute the conditional pdf of X given that the coin shows heads, $f_{X|\text{Heads}}(x)$, using the continuous Bayes formula:

$$f_{X|\text{Heads}}(x) = \frac{f_X(x) P(\text{Heads}|X = x)}{P(\text{Heads})}.$$

(3) Using the conditional pdf, compute $E[X|\text{Heads}]$.
(4) Suppose instead the coin shows tails. Compute $f_{X|\text{Tails}}(x)$ and $E[X|\text{Tails}]$.

Problem 2. Two independent machines, A and B, each require a random time to complete a task. Let X and Y denote these times, both exponentially distributed with rate $\lambda > 0$.

Define the total completion time and the relative share of A in the total time by

$$Z = X + Y, \qquad R = \frac{X}{X + Y}.$$

(1) Determine the range of possible values of *the pair* (Z, R) as a region on the plane (see textbook, Sec.8.1).
(2) Derive the joint pdf $f_{Z,R}(z, r)$ using the transformation from (X, Y) to (Z, R).
(3) Find the marginal pdfs $f_Z(z)$ and $f_R(r)$, and identify the distributions of Z and R.
(4) Show that Z and R are independent, and explain intuitively why this makes sense in the context of the problem.
(5) Compute $E[Z]$, $\text{Var}(Z)$, and $E[R]$.

Problem 3. A well-shuffled standard deck of 52 cards contains 4 aces. You draw 3 cards at random without replacement and let X be the number of aces among the three cards.

(1) Compute the unconditional expectation $E[X]$.
(2) Let A be the event “at least one ace is drawn” (i.e. $A = \{X \ge 1\}$). Compute the conditional expectation $E[X|A]$.
(3) Let B be the event “the first card drawn is an ace”. Compute $E[X|B]$.

(4) Compare $E[X|A]$ and $E[X|B]$. Which is larger and why? Give a short intuitive explanation (you may also support your answer with the numerical values).

Problem 4. Let X and Y be continuous random variables with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Verify that $f_{X,Y}(x,y)$ is a valid joint pdf.
(2) Let $g(X,Y) = X^2Y + Y^2$. Compute the expectation

$$E[g(X,Y)] = \int_0^1 \int_0^1 g(x,y) f_{X,Y}(x,y)\, dx\, dy.$$

(3) Let $h(X,Y) = X + Y$. Compute $E[h(X,Y)^2]$.
(4) Compute $\text{Cov}(X,Y)$ and $\text{Var}(X+Y)$ using your results from above.

Problem 5. Two fair dice are rolled. Let X be the outcome of the first die and Y the outcome of the second die.

(1) Let $g(X,Y) = X \cdot Y + X$. Compute the expectation
(2) Let $h(X,Y) = X + Y$. Compute $E[h(X,Y)^2]$.
(3) Compute $\text{Cov}(X,Y)$ and $\text{Var}(X+Y)$.
(4) Let $I = \mathbf{1}(X = Y)$ be the indicator that both dice show the same number. Compute $E[I]$ and $E[g(X,Y)|I = 1]$.