

ENEE324-03: Engineering Probability

Midterm Examination

March 12, 2015

Each problem is worth 10 points

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- Please be precise and rigorous in your statements, and show all your calculations. Please write neatly and legibly. Be sure to **Print Your Name!**
- This is a closed book exam, but you are allowed up to two 8.5× 11 pages of notes. No calculators please! Good luck!

Problem 1.

A number is chosen at random from the set of integers $\{1, 2, \dots, 200\}$. What is the probability that it is divisible by 3 or 5 (i.e., either 3 or 5 or both)?

Solution: Let $A = \{i \in (1..200) : i \text{ divisible by } 3\}$, $B = \{i \in (1..200) : i \text{ divisible by } 5\}$, then $|A| = \lfloor 200/3 \rfloor = 66$; $B = \lfloor 200/5 \rfloor = 40$; $|AB| = \lfloor 200/15 \rfloor = 13$. Then

$$p = \frac{1}{200}(66 + 40 - 13) = \frac{46.5}{100} = 0.465.$$

Problem 2.

There are two coins one of which is genuine and the other counterfeit. The real one is fair, and the false one has the probability of coming up heads 60%. You get to choose a coin uniformly and randomly among the two coins, and upon having chosen, flip it 3 times. The observed outcome is HHH. What is the probability that the chosen coin is false?

Solution: Let $A = \{HHH\}$, and let F be the event that the coin is false. We have

$$P(F|A) = \frac{P(A|F)P(F)}{P(A)} = \frac{0.216 \cdot 1/2}{(0.216 + 0.125) \cdot 1/2} = \frac{0.216}{0.341} \approx 0.633.$$

Problem 3.

Roll two dice. Consider the events $A = \text{"The first die is odd"}$, $B = \text{"The second die is odd"}$, $C = \text{"The sum is odd"}$. Are A , B , and C pairwise independent? Are they jointly independent?

Solution: $P(A) = 1/2$, $P(B) = 1/2$

$P(C) = P\{\text{all pairs of the form (odd, even) and (even, odd), where the entries are between 1 and 6}\}$

$$= P\left\{\begin{pmatrix} (1, 2) & (1, 4) & (1, 6) \\ (2, 1) & (2, 3) & (2, 5) \\ (3, 2) & (3, 4) & (3, 6) \\ (4, 1) & (4, 3) & (4, 5) \\ (5, 2) & (5, 4) & (5, 6) \\ (6, 1) & (6, 3) & (6, 5) \end{pmatrix}\right\} = 1/2.$$

$$P(AB) = P(A)P(B), P(AC) = P(A)P(C), P(BC) = P(B)P(C)$$

$$P(ABC) = P(\{\emptyset\}) = 0 \neq P(A)P(B)P(C)$$

Answer: Yes, No.

Problem 4.

Each of the three players has a $(p, 1 - p)$ coin. In each round of the game, they toss their coins simultaneously; if one of them gets an outcome different from the other two, he is declared the winner. If all the outcomes are equal, they move to the next round.

(a) Find the probability that the winner emerges in less than n rounds.

(b) Find the expectation and the variance of the number of rounds played before they have the winner.

Solution: (a) Let $q = 1 - p$. The probability of having a winner in any given round is

$$3p^2q + 3q^2p = 3pq(p + q) = 3pq.$$

We have a geometric random variable X with success probability $3pq$. The probability of finishing the game in $i < n$ rounds is

$$P(X < n) = 1 - P(X \geq n) = \sum_{i=1}^{n-1} (3pq)(1-3pq)^{i-1} = 3pq \frac{1 - (1-3pq)^{n-1}}{1 - (1-3pq)} = 1 - (1-3pq)^{n-1}.$$

(b) For a geometric RV we have $EX = \frac{1}{3pq}$, $\text{Var}(X) = \frac{1-3pq}{9p^2q^2}$.

(If “before” in question (b) is read to mean not including the winning round, then the number of rounds is $Z = X - 1$, and $EZ = EZ - 1$, $\text{Var}(Z) = \text{Var}(X)$. Either answer is fine.)

Problem 5.

Let K be a random variable with PMF

$$p_K(k) = \begin{cases} k^2/a & \text{if } k = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{o/w} \end{cases}$$

(a) Find a and EK .

(b) What is the pmf of the RV $L = (K - EK)^2$?

(c) Find the variance of K using the result of part (b). Then find the variance of K using the formula $\text{Var}(K) = \sum_k p_K(k)(k - E[K])^2$.

Solution: (a) By normalization, $a = \sum_{k=-3}^3 k^2 = 28$. $EK = 0$ because K is symmetric (i.e., $p_K(-k) = p_K(k)$).

(b)

$$p_L(k) = \begin{cases} p_K(-\sqrt{k}) + p_K(\sqrt{k}) = k/14 & \text{if } k = 1, 4, 9 \\ 0 & \text{o/w} \end{cases}$$

(c) Using Part (b) we have $\text{Var}(K) = EL = 1/14 + 4 \cdot 4/14 + 9 \cdot 9/14 = 98/14 = 7$. Also

$$\text{Var}(K) = \sum_{k=-3}^3 p_K(k)(k - E[K])^2 = 9 \cdot (9/14) + 4 \cdot (4/14) + (1/14) = 7$$

(substituting $EK = 0$ and the values of $p_K(k)$).