**ENEE324. Midterm examination 1, 10/20/2025.** Instructor: A. Barg

- The paper contains 5 problems. Each problem is 10 points. Max score=50 points
- Your answers should be justified.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.
- DO NOT copy the problem statement into your paper
- Simplify your calculations as much as you can. Perform divisions, multiplications, cancellations etc.
- SIGN YOUR NAME!

**Problem 1.** A website requires a 6-character password made of uppercase letters A,B,C and digits 0 and 1.

(a) If an attacker guesses passwords at random, what is the probability that he succeeds at least once after making 100 tries?

(b) What is the minimum number of guesses the attacker must make for the expected number of successes to exceed 0.5?

(c) Choose a random password. What is the probability that it is formed only of the letters A,B,C and does not include digits?

**Solution:** (a) The probability $p$ of success in a single try is $p = \frac{1}{5^6} = \frac{1}{15625}$. Let $X \sim \text{Binom}(100, p)$. The probability in question is $P(X \geq 1) = 1 - P(X = 0) = 1 - (1-p)^{100} \approx 0.00638$.

(b) The expected number of successes is $np$. Taking $np = 0.5$, we have $n = \lceil \frac{15625}{2} \rceil = 7813$.

(c) $\frac{3^6}{5^6} = \frac{729}{15625} = 0.0467$

**Problem 2.** A company manufactures two kinds of batteries, Type I and Type II.

- $25\%$ of the batteries produced are Type I, and $75\%$ are Type II.
- Type I batteries last a random number $Y_I$ of months, with pmf

$$p_{Y_I}(k) = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}, \qquad k = 1, 2, \ldots$$

- Type II batteries last a random number $Y_{II}$ of months, with pmf

$$p_{Y_{II}}(k) = \frac{1}{2}\left(\frac{1}{2}\right)^{k-1}, \qquad k = 1, 2, \ldots$$

A single battery is selected at random from the production line. Let $Y$ denote its lifetime.

(1) Validity check: Show that $p_{Y_I}$ and $p_{Y_{II}}$ are valid pmfs.
(2) Mixture distribution: Find the pmf of $Y$ (justification required).
(3) Expected lifetime: Compute $\mathbb{E}[Y]$.
(4) Reliability after 6 months: Compute $\Pr(Y \geq 6)$.
(5) Conditional classification: Suppose a randomly chosen battery is still working after 6 months (at the end of the 6th month). Compute the probability that it is of Type I conditional on this information.

**Solution:** Random variables $Y_I$ and $Y_{II}$ are FIRST SUCCESS random variables.

(1) $\sum_{k=1}^{\infty} p(1-p)^{k-1} = p\sum_{k=0}^{\infty}(1-p)^k = p\frac{1}{1-(1-p)} = 1$ for any $p \in (0,1)$.

(2) First, note that $P(\text{FS}(p) \geq n) = \sum_{k=n}^{\infty} pq^{k-1} = \frac{p}{q}\sum_{k=n}^{\infty} q^k = q^{n-1}$. Then, using LOTP, for any $k \geq 1$,

$$P(Y = k) = P(Y = k|I)P(I) + P(Y = k|II)P(II) = 0.25p_{Y_I}(k) + 0.75p_{Y_{II}}(k)$$
$$= \frac{1}{12}\Big(\frac{2}{3}\Big)^{k-1} + \frac{3}{8}\Big(\frac{1}{2}\Big)^{k-1}$$

(3) The expectation of $\text{FS}(p)$ equals $\frac{1}{p}$ for all $0 < p < 1$. By linearity of expectation, we obtain $\mathsf{E}(Y) = 0.25\mathsf{E}Y_I + 0.75\mathsf{E}Y_{II} = \frac{3}{4} + \frac{3}{2} = \frac{9}{4}$.

(4) $P(Y \geq 6) = P(Y \geq 6|I)P(I) + P(Y \geq 6|II)P(II)$. As in HW4,

$$P(Y \geq 6) = \frac{1}{4}\left(\frac{2}{3}\right)^5 + \frac{3}{4}\left(\frac{1}{2}\right)^5 = \frac{1753}{128 \cdot 243} \approx 0.0563.$$

(5) The Bayes formula gives $P(I|Y \geq 6) = \frac{P(Y\geq 6|I)P(I)}{P(Y \geq 6)} = \frac{(\frac{2}{3})^5 \cdot 0.25}{0.0563} \approx 0.584.$

**Problem 3.** A company buys sensors from three suppliers: A, B, and C.

- $40\%$ of the sensors come from supplier A,
- $35\%$ from supplier B,
- $25\%$ from supplier C.

The probability that a sensor is defective depends on the supplier:

$$P(\text{defective} \mid A) = 0.02, \qquad P(\text{defective} \mid B) = 0.05, \qquad P(\text{defective} \mid C) = 0.10.$$

A sensor is selected at random from the warehouse.

(1) Compute $P(\text{defective})$.
(2) Suppose the sensor is found to be defective. Compute the posterior probabilities

$$P(A \mid \text{defective}), \quad P(B \mid \text{defective}), \quad P(C \mid \text{defective}).$$

(3) Suppose the sensor is not defective. Compute $P(C \mid \text{not defective})$.
(4) Inspecting one sensor costs \$1, and repairing a defective sensor found during inspection saves the company \$40 (for type A), \$50 (for type B), and \$60 (for type C) in expected future costs. The company will inspect only sensors from one supplier chosen in advance. Which supplier should the company inspect to maximize expected net benefit per inspected sensor? (Compute expected benefit per inspected sensor for each supplier and state the best choice.)

**Solution:** We are given that $P(A) = 0.4, P(B) = 0.35, P(C) = 0.25$. Below we write d for defective.
1.

$$\begin{aligned}
P(d) &= P(d, A) + P(d, B) + P(d, C) \\
&= P(d|A)P(A) + P(d|B)P(B) + P(d|C)P(C) \\
&= 0.02 \cdot 0.4 + 0.05 \cdot 0.35 + 0.1 \cdot 0.25 \\
&= 0.008 + 0.0175 + 0.025 = 0.0505.
\end{aligned}$$

2. Using the Bayes formula,

$$P(A|d) = \frac{P(d|A)P(A)}{P(d)} = \frac{0.008}{0.0505} \approx 0.158.$$

$$P(B|d) = \frac{0.0175}{0.0505} \approx 0.347, \quad P(C|d) = \frac{0.025}{0.0505} \approx 0.495$$

As a sanity check, these three numbers add to 1 (up to the rounding errors).
3.

$$P(C|\bar{d}) = \frac{P(\bar{d}|C)P(C)}{P(\bar{d})} = \frac{0.9 \cdot 0.25}{0.9495} = \frac{0.225}{0.9495} \approx 0.237.$$

4. If we check, we always pay \$1. If the sensor is defective, which happens with probability $P(d|S)$, we save \$ 40 or 50 or 60. Thus, the expected gain/loss from inspecting a sensor from supplier $S$ is $-1 + (\text{savings}) \cdot P(d|S)$. Denote $G_S$ to be the gain from inspecting sensors of type $S$. We compute

$$G_A = -1 + 40P(d|A) = -0.2, \; G_B = -1 + 50P(d|B) = 1.5, \; G_C = -1 + 60P(d|C) = 5.$$

Thus, the best choice is supplier C.

**Problem 4.** A particle starts at position 0 on the integer line. At each step it moves

- one unit to the right with probability $p = \frac{2}{3}$,

- one unit to the left with probability $1-p=\frac{1}{3}$.

Let $S_n$ denote the position of the particle after $n$ steps.

(1) Find $P(S_3=1)$.
(2) Compute $P(S_4=0)$.
(3) Compute $\mathsf{E}[S_n]$.
(4) Compute $P(S_{10}\geq 0)$

Hint: Introduce a random variable $X_n$ equal to the number of steps to the right after $n$ steps of the process, and express $S_n$ in terms of $X_n$.

**Solution:** 1. We note that $X_n = \text{Binom}(n,2/3)$ and $S_n = 2X_n - n$. Thus,

$$P(S_3=1)=P(2X_3=4)=P(X_3=2)=\binom{3}{2}p^2(1-p)=3\cdot\frac{4}{9}\cdot\frac{1}{3}=\frac{4}{9}.$$

2. To end up at 0 after an even number n of steps, we must make $n/2$ steps in each of the two directions.

$$P(S_4=0)=\binom{4}{2}p^2(1-p)^2=6\cdot\frac{4}{9}\cdot\frac{1}{9}=\frac{8}{27}$$

3.

$$\mathsf{E}\,S_n=\mathsf{E}\,(2X_n-n)=2\mathsf{E}\,X_n-n=2np-n=\frac{n}{3},$$

which makes sense because $p-(1-p)=\frac{1}{3}$, i.e., the walk is $\frac{1}{3}$ more likely to drift right than left.

4.

$$P(S_{10}\geq 0)=P(2X_{10}\geq 10)=P(X_{10}\geq 5)=\sum_{k=5}^{10}\binom{10}{k}(2/3)^k(1/3)^{10-k}\approx 0.923.$$

**Problem 5.** A machine produces components whose lifetime (in years) is modeled by the pmf

$$p_X(k)=c\cdot\frac{1}{k(k+1)},\qquad k=1,2,3,\ldots$$

where $c$ is a normalizing constant.

(1) Find $c$ (rewrite the summation term as a difference of two fractions)
(2) Compute $P(X\geq 3)$.
(3) What is the expectation $\mathbb{E}[X]$?

**Solution:** (1) The answer is suggested by writing

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=\sum_{i=1}^{\infty}\Big(\frac{1}{k}-\frac{1}{k+1}\Big).$$

This looks like 1 because all the terms except the first one cancel. This isn't a formal argument because if it was, then the sum $1-1+1-1+1-\ldots$ would have a finite value, which it does not. To show this formally, we recall that by definition, $\sum_{k=1}^{\infty}a_k=\lim_{n\to\infty}\sum_{k=1}^{n}a_k$, and write

$$\sum_{k=1}^{n}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)=1-\frac{1}{n+1}.$$

As $n\to\infty$, the sequence of these sums converges to 1. This shows that $c=1$ by the normalization property of probability.

(2)

$$P(X\geq 3)=1-P(X\leq 2)=1-\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$$

(3)

$$EX = \sum_{k=1}^{\infty} k p_X(k) = \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty$$

Thus, $X$ has infinite expectation.