ENEE324. Midterm examination 1, 3/11/2026. Instructor: A. Barg

- The paper contains 5 problems. Each problem is 10 points. Max score=50 points
- Your answers should be justified.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.
- DO NOT copy the problem statement into your paper
- Simplify your calculations as much as you can. Perform divisions, multiplications, cancellations etc.
- SIGN YOUR NAME!

Problem 1 (The Bayes formula)**.** A university uses an automated system to detect plagiarism in submitted essays. Historically, about 2% of essays contain plagiarized material.

If an essay actually contains plagiarism, the system flags it with probability 90%. However, it also incorrectly flags 5% of essays that do not contain plagiarism.

(a) An essay is flagged by the system. What is the probability that it actually contains plagiarism?

(b) Suppose the essay is checked independently by two different plagiarism–detection systems of the same type, and both systems flag it. What is now the probability that the essay actually contains plagiarism?

(c) Suppose the essay is checked independently by two such systems, the first system flags it but the second does not. What is the probability that the essay actually contains plagiarism?

Start with introducing notation for the events described in the statement and writing the conditional probabilities from the verbal description.

Solution:

Let $\mathcal{P}$ denote the event that an essay contains plagiarism and $\mathcal{F}$ the event that a system flags the essay. We are given $P(\mathcal{P}) = 0.02, P(\mathcal{P}^c) = 0.98$, $P(\mathcal{F}|\mathcal{P}) = 0.90, P(\mathcal{F}|\mathcal{P}^c) = 0.05$.

(a) By Bayes' formula,

$$P(\mathcal{P}|\mathcal{F}) = \frac{P(\mathcal{F}|\mathcal{P})P(\mathcal{P})}{P(\mathcal{F}|\mathcal{P})P(\mathcal{P}) + P(F|\mathcal{P}^c)P(\mathcal{P}^c)} = \frac{0.90 \cdot 0.02}{0.90 \cdot 0.02 + 0.05 \cdot 0.98} = \frac{18}{67} \approx 0.269.$$

(b) Let $\mathcal{F}_1, \mathcal{F}_2$ be the events that the first and second systems flag the essay. Assuming the systems act independently conditional on whether plagiarism is present,

$$P(\mathcal{F}_1\mathcal{F}_2|\mathcal{P}) = 0.9^2 = 0.81, \qquad P(\mathcal{F}_1\mathcal{F}_2|\mathcal{P}^c) = 0.05^2 = 0.0025.$$

Applying Bayes' formula,

$$\begin{aligned} P(\mathcal{P}|\mathcal{F}_1\mathcal{F}_2) &= \frac{P(\mathcal{F}_1\mathcal{F}_2|\mathcal{P})P(\mathcal{P})}{P(\mathcal{F}_1\mathcal{F}_2|\mathcal{P})P(\mathcal{P}) + P(\mathcal{F}_1\mathcal{F}_2|\mathcal{P}^c)P(\mathcal{P}^c)} \\ &= \frac{0.81 \cdot 0.02}{0.81 \cdot 0.02 + 0.0025 \cdot 0.98} = \frac{0.0162}{0.0162 + 0.00245} \approx 0.869. \end{aligned}$$

(c) Now consider the event $\mathcal{F}_1\mathcal{F}_2^c$ (the first system flags the essay but the second does not).
Conditional probabilities:

$$P(\mathcal{F}_1\mathcal{F}_2^c|\mathcal{P}) = 0.9 \cdot 0.1 = 0.09,$$
$$P(\mathcal{F}_1\mathcal{F}_2^c|\mathcal{P}^c) = 0.05 \cdot 0.95 = 0.0475.$$

Again by Bayes' formula,

$$P(\mathcal{P}|\mathcal{F}_1\mathcal{F}_2^c) = \frac{0.09 \cdot 0.02}{0.09 \cdot 0.02 + 0.0475 \cdot 0.98} = \frac{0.0018}{0.0018 + 0.04655} \approx 0.037.$$

Problem 2 (Straightforward counting)**.** A system generates an 8-character access code. Each character is chosen independently from the set

$$\{A, B, C, 0, 1\}.$$

(a) What is the probability that a randomly generated code contains exactly three digits?

(b) What is the probability that the code contains at least one digit?
(c) An attacker guesses codes uniformly at random, using independent choices without keeping record of the previous guesses. If the true code is fixed but unknown, what is the probability that the attacker succeeds at least once after n independent guesses?
(d) An attacker guesses codes as in Part (c). Let X be the number of successful guesses in n attempts. What is the probability of k successes? Which distribution law does this probability represent?
(e) Suppose that two attackers simultaneously guess the code, so each attempt is two independent guesses. Call the attempt a **success** if at least one of the guesses is correct. Let Y be the random number of successes in n attempts. What is the probability that there are k successes in n attempts?

Solution:
Each position of the code is chosen independently from the set $\{A, B, C, 0, 1\}$, so there are 5^8 possible codes in total. Digits are $\{0, 1\}$ (2 choices) and letters are $\{A, B, C\}$ (3 choices).
(a) We want exactly three digits among the eight positions. The number of such codes is $\binom{8}{3}2^3 3^5$. Therefore

$$P(\text{exactly 3 digits}) = \frac{\binom{8}{3}2^3 3^5}{5^8} \approx 0.279.$$

(b) The probability that the code contains at least one digit equals

$$1 - P(\text{no digits}) = 1 - \left(\frac{3}{5}\right)^8 \approx 0.983.$$

(c) The probability that the attacker succeeds at least once is

$$1 - \left(1 - \frac{1}{5^8}\right)^n.$$

(d) Let X be the number of successful guesses in N attempts. Then $X \sim \text{Binomial}\big(N, p = 1/5^8\big)$, and $P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$.
(e) The probability that an attempt is a failure is $(1-p)^2$, so the probability of success is $r = 1-(1-p)^2$ and $Y \sim \text{Binom}(n, r)$.

Problem 3 (Recall HW3)**.** A deck contains 6 red cards and 8 blue cards. Cards are drawn one at a time without replacement until all red cards have appeared. Let T be the number of draws required.
(a) What are the possible values of T?
(b) For each value k of T identified in part (a), express $P(T = k)$ in terms of binomial coefficients.
(c) Compute $P(T = 14)$ (give a number).
(d) Suppose that among the first 7 draws exactly 4 cards are blue. Compute the probability that $T = 10$ (give a number).

Solution: There are 6 red and 8 blue cards, total 14 cards. Let T be the position of the last red card.
(a) At least 6 draws are needed to see all red cards, and at most 14, so

$$T \in \{6, 7, \dots, 14\}.$$

(b) For $T = k$, the kth card must be red and among the first $k-1$ cards there must be exactly 5 red cards. Thus

$$P(T = k) = \frac{\binom{6}{5}\binom{8}{k-6}}{\binom{14}{k-1}} \cdot \frac{1}{15-k}, \quad k = 6, 7, \dots, 14.$$

(c) For $T = 14$, the last card is red. Among the first 13 cards, choose 5 red cards and 8 blue cards:

$$P(T = 14) = \frac{\binom{6}{5}\binom{8}{8}}{\binom{14}{13}} = \frac{6}{14} = \frac{3}{7}.$$

(d) Given 3 red and 4 blue in the first 7 cards, the remaining cards contain 3 red and 4 blue. For $T = 10$, the 8th, 9th, and 10th cards must be the remaining red cards:

$$P(T = 10|3 \text{ red in first } 7) = \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5} = \frac{1}{35}.$$

Problem 4 (Definition of indepdence)**.** A factory produces electronic components in two machines, Machine A and Machine B. Machine A produces 60% of the components, and Machine B produces 40%. The probability that a component is defective is 0.02 if it comes from Machine A, and 0.05 if it comes from Machine B.

Let D be the event that a randomly selected component is defective, and let M_A and M_B be the events that the component comes from Machine A or Machine B, respectively.

(a) Are the events D and M_A independent? Justify your answer.
(b) A quality inspector tests two components chosen independently at random. Let D_1 and D_2 be the events that the first and second components are defective. Are D_1 and D_2 independent? Explain.
(c) Suppose the inspector knows that both tested components are from Machine B. Let D_1 and D_2 be the events that the first and second components are defective. Are D_1 and D_2 independent given that both come from Machine B? Explain.

Solution: We are given

$$P(M_A) = 0.6, \quad P(M_B) = 0.4, \quad P(D|M_A) = 0.02, \quad P(D|M_B) = 0.05.$$

(a) Independence of D and M_A. Compute $P(D)$ using the law of total probability:

$$P(D) = P(D|M_A)P(M_A) + P(D|M_B)P(M_B) = 0.02 \cdot 0.6 + 0.05 \cdot 0.4 = 0.032.$$

Then

$$P(D, M_A) = P(D|M_A)P(M_A) = 0.02 \cdot 0.6 = 0.012 \neq P(D)P(M_A) = 0.032 \cdot 0.6 = 0.0192.$$

Hence D and M_A are *not independent*.
Even simpler: $P(D) \neq P(D|M_A)$ (both computed above).
(b) Independence of D_1 and D_2.
The two components are chosen independently from the factory. Defectiveness depends only on the machine the component comes from. Since the components are chosen independently and machines operate independently, D_1 and D_2 are *independent*. Formally, $P(D_1|D_2) = P(D_1)$ since different components are produced independently.
(c) Conditional independence given both from Machine B
Given that both components come from Machine B, D_1 and D_2 depend only on whether each component is defective independently (since defects occur independently for each component). Thus D_1 and D_2 are *conditionally independent* given that both are from Machine B:

$$P(D_1 D_2 | M_B^{(1)} M_B^{(2)}) = P(D_1|M_B)P(D_2|M_B) = 0.05^2 = 0.0025.$$

Problem 5 (Recall HW2)**.** Two players, Alice and Bob, play a game using a biased coin that lands heads with probability p. They take turns flipping the coin, with Alice going first. The first player to get a head wins.

(a) What is the probability that Alice wins on her *second* turn (i.e., after both have flipped once without a head)?
(b) What is the probability that Alice wins on her kth turn for $k = 3, 4, 5, \dots$?
(c) Using the result of Part (b), compute the total probability P_A that Alice wins the game in terms of p.

(d) Use the fact that at the beginning of the second round (both A and B failed once), the game resets to its starting state to compute P_A in a different way. Make sure you obtain the same answer as in Part (c).

Solution:

(a) Both Alice and Bob must fail on the first turn (probability $(1-p)^2$), then Alice must succeed on her second turn. Hence $P(\text{Alice wins second turn}) = (1-p)^2 \cdot p$.

(b) On each "round" (Alice flip + Bob flip) in which no one gets heads, the probability is $(1-p)^2$. Thus, for $k \geq 1$,

$$P(\textbf{Alice wins on round } k) = (1-p)^{2(k-1)} p.$$

(c) Total probability Alice wins: The game starts with round 1 and continues indefinitely. Altogether, these quantities form a geometric series:

$$P_A = \sum_{k=0}^{\infty} p\big((1-p)^2\big)^k = \frac{p}{1-(1-p)^2} = \frac{p}{2p-p^2} = \frac{1}{2-p}.$$

(d) We have

$$P_A = p + (1-p)^2 P_A.$$

Solving for P_A, we again obtain $\frac{1}{2-p}$.