ENEE324-03: Engineering Probability

Midterm Examination #2 **April 28, 2015**

Max total 55 pts. Instructor: A. Barg

• Please be precise and rigorous in your statemens, and show all your calculations. Please write neatly and legibly. Be sure to **Print Your Name!**

• This is a closed book exam, but you are allowed up to two 8.5× 11 pages of notes. No calculators please! Good luck!

Problem 1 (10pts)
Let the joint PMF of discrete RVs be given by

$$p_{XY}(x,y) = \begin{cases} \frac{1}{25}(x^2+y^2) & \text{if } x = 1,2,\ y = 0,1,2 \\ 0 & \text{otherwise.} \end{cases}$$

Find $p_{X|Y}(x|y), P(X=2|Y=1)$ and $E(X|\{Y=1\})$.

Solution:

$$p_Y(y) = p_{XY}(1,y) + p_{XY}(2,y) = \frac{1}{25}(2y^2+5), \quad y = 0,1,2.$$

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)} = \frac{x^2+y^2}{2y^2+5}, \quad y = 0,1,2; x = 1,2.$$

$$P(X=2|Y=1) = p_{X|Y}(2|1) = 5/7.$$

$$E(X|\{Y=1\}) = 1 \cdot p_{X|Y}(1|1) + 2 \cdot p_{X|Y}(2|1) = \frac{2}{7} + 2 \cdot \frac{5}{7} = \frac{12}{7}.$$

Problem 2 (10 pts)
Let X be an RV with PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{o/w} \end{cases}$$

for some $\lambda > 0$. Find the PDF of the RV $Y = X^{2/3}$.

Solution: We have

$$F_Y(y) = P(Y \le y) = P(X^{2/3} \le y) = P(X \le y^{3/2}) = \int_0^{y^{3/2}} f_X(x)dx = 1 - e^{-\lambda y^{3/2}}, y \ge 0.$$

$$f_Y(y) = F_Y'(y) = \frac{3\lambda}{2}\sqrt{y}e^{-\lambda y\sqrt{y}}, \quad y \ge 0.$$

Problem 3 (10pts)
The joint pdf of RVs X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{1}{16\pi} & \text{if } x^2+y^2 \le 16 \\ 0 & \text{o/w} \end{cases}$$

(a) Find $f_X(x)$, $E[X|Y]$

(b) Let $R = \sqrt{X^2+Y^2}$. Find $f_R(r)$ (hint: the pair X, Y is jointly uniformly distributed in the circle).

(c) Are X and Y independent?

Solution: (a) $f_X(x) = \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} f_{XY}(x,t)dt = \frac{\sqrt{16-x^2}}{8\pi}, -4 \le x \le 4$. By symmetry we have $f_Y(y) = \frac{\sqrt{16-y^2}}{8\pi}, -4 \le y \le 4$ and $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{2\sqrt{16-y^2}}$, $-\sqrt{16-y^2} \le x \le \sqrt{16-y^2}$. Then for any $y, -4 \le y \le 4$

$$E[X|Y=y] = \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \frac{xdx}{2\sqrt{16-y^2}} = 0$$

and $E[X|Y] = 0$ with probability one (this is of course clear by symmetry without any computations).

(b)

$$F_R(r) = \begin{cases} 0 & \text{if } r \le 0 \\ P(R \le r) = P(X^2+Y^2 \le r^2) = \frac{\pi r^2}{16\pi} = (\frac{r}{4})^2 & \text{if } 0 \le r \le 4 \\ 1 & \text{if } r \ge 4 \end{cases},$$

and $f_R(r) = F_R'(r) = r/8$ if $0 \le r \le 4$ and $= 0$ otherwise.

(c) From the definition and the PDFs computed above it is immediate that X and Y are not independent.

Problem 4 (15pts)
Let $X \sim \text{Unif}[0,1], Y \sim \exp(\lambda)$ be independent RVs. Let $Z = X + Y$.

(a) (10pts) Find the PDF $f_Z(z)$ and the CDF $F_Z(z)$.

(b) (5pts) Find $E(Z^2)$ (Hint: This question can be solved independently of the solution of part (a)).

Solution: (a)

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy.$$

If $0 \le z \le 1$, then $f_X(z-y) \ne 0$ for $0 \le y \le z$. If $z \ge 1$ then $f_X(z-y) \ne 0$ for $z-1 \le y \le z$. We obtain

$$f_Z(z) = \begin{cases} \int_0^z \lambda e^{-\lambda y}dy = 1 - e^{-\lambda z}, & 0 \le z \le 1 \\ \int_{z-1}^z \lambda e^{-\lambda y}dy = e^{-\lambda(z-1)} - e^{-\lambda z}, & z \ge 1. \end{cases}$$

$$F_Z(z) = \begin{cases} \int_0^z (1 - e^{-\lambda t}dt = z + \frac{1}{\lambda}(e^{-\lambda z} - 1) & 0 \le z \le 1 \\ \int_0^z f_Z(t)dt = F_Z(1) + \int_1^z (e^{-\lambda(t-1)} - e^{-\lambda t})dt & \\ \quad = F_Z(1) + (-\frac{1}{\lambda}\Big)(e^{-\lambda(t-1)} - e^{-\lambda t})|_1^z & \\ \quad = 1 + \frac{1}{\lambda}(e^{-\lambda} - 1) - \frac{1}{\lambda}(e^{-\lambda(z-1)} - e^{-\lambda z}) + \frac{1}{\lambda}(1 - e^{-\lambda}) & \\ \quad = 1 - \frac{1}{\lambda}(e^{-\lambda(z-1)} - e^{-\lambda z}), & z \ge 1. \end{cases}$$

(b) $EX^2 = 1/3, EY^2 = \int_0^\infty y^2 e^{-\lambda y}/\lambda dy = -y^2 e^{-\lambda y}|_0^\infty + 2\int_0^\infty y e^{-\lambda y}dy = 2/\lambda^2$ (the last step because $EY = 1/\lambda$). Then

$$E[X^2 + 2XY + Y^2] = \frac{1}{3} + \frac{2}{\lambda^2} + 2(EX)(EY)(\text{by independence}) = \frac{1}{3} + \frac{2}{\lambda^2} + \frac{1}{\lambda}.$$

Problem 5 (10pts)

Let X be an RV with pdf $f_X(x) = x\ e^{-x}$ where $x > 0$. Calculate:

(1) The transform $M_X(s)$ for X.
(2) The moment $E[X^n]$ using $M_X(s)$ above.
Hint: the following relation could be useful

$$\sum_{k=0}^{\infty}(k+1)x^k = \frac{1}{(1-x)^2}.$$

Solution:

(1) $M_X(s) = \int\limits_0^\infty xe^{-x}e^{sx}dx = \int\limits_0^\infty xe^{-(1-s)x}dx = \frac{1}{(1-s)^2}$, $s < 1$

(2) $M_X(s) = (1-s)^{-2} = \sum\limits_{k=0}^\infty (k+1)s^k = \sum\limits_{k=0}^\infty \frac{(k+1)!}{k!}s^k$

(Alternatively, $M_X(s) = (1-s)^{-2} = \sum\limits_{k=0}^\infty \binom{-2}{k}(-s)^k = \sum\limits_{k=0}^\infty \frac{(k+1)!}{k!}s^k$, $s < 1$) We have,

$$\sum_{k=0}^\infty M_X^{(k)}(0)\frac{s^k}{k!} = \sum_{k=0}^\infty (k+1)s^k.$$

Taking $k = n$, we obtain $\frac{M_X^{(n)}(0)}{n!} = n+1$ and

$$E[X^n] = M_X^{(n)}(0) = (n+1)!.$$