ENEE324. Midterm examination 2, 11/19/2025. Instructor: A. Barg

- The paper contains 5 problems. Each problem is 10 points. Max score=50 points
- Your answers should be justified.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.
- DO NOT copy the problem statement into your paper
- Simplify your calculations as much as you can. Perform divisions, multiplications, cancellations etc.
- SIGN YOUR NAME!

Problem 1. Customers arrive at a coffee shop according to a Poisson process with rate $\lambda = 3$ customers per hour.

(1) Suppose the time count starts at some fixed hour, call it hour 0. What is the probability that exactly 5 customers arrive in the first 2 hours?

(2) Let T_1 be the time until the first customer arrives.

  (a) Find the pdf of T_1. For this, use the fact that $P(T_1 \le t) = 1 - P(T_1 \ge t)$, where the event in the last $P(\cdot)$ means that there were no arrivals in $[1, t]$.

  (b) Compute $P(T_1 > 20 \text{ minutes})$.

(3) Suppose that exactly 4 customers arrive in the first hour.

  (a) Given this information, what is the probability that exactly 2 of them arrived in the first 30 minutes?

  (b) Given this information, what is the expected number of customers in the second half-hour? What is the expected number of customers in the second half-hour not conditioned on any other information?

Solution. (1) For a Poisson process $N(t)$ with rate λ and any $k = 0, 1, \dots,$

$$P(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}.$$

Here $\lambda t = 3 \cdot 2 = 6$ and $k = 5$. Thus

$$P(N(2) = 5) = e^{-6}\frac{6^5}{5!} \approx 0.16.$$

(2) (a) For a Poisson process with rate λ, the time to the first event is exponential with rate λ. This is easy to compute as described in the hint for (2a): $P(T_1 \ge t) = P(N(T_1) = 0) = e^{-\lambda t}$, so $P(T_1 \le t) = 1 - e^{-\lambda t}$, which is the CDF of Exp(λ). Then $f_{T_1}(t) = \lambda e^{-\lambda t}, t \ge 0$. In our case, $\lambda = 3$, so $f_{T_1}(t) = 3e^{-3t}$.

(b) Twenty minutes is $1/3$ hour. As above, $P(T_1 > t) = e^{-\lambda t}$, so

$$P\big(T_1 > \tfrac{1}{3}\big) = e^{-3\cdot(1/3)} = e^{-1} \approx 0.368.$$

(3) (a) The number of customers that fall in the first half-hour (length 0.5) is $\mathrm{Bin}(4, 0.5)$. Therefore

$$P\{\text{exactly 2 in first } 0.5 \text{ h} \mid N(1) = 4\} = \binom{4}{2}(0.5)^2(0.5)^2 = \binom{4}{2}(0.5)^4 = 6 \cdot \frac{1}{16} = \frac{3}{8}.$$

(b) Conditioned on $N(1) = 4$, the count in $[0.5, 1]$ has distribution $\mathrm{Bin}(4, 0.5)$. Its mean is $4 \cdot 0.5 = 2$. Without the conditioning, the number of arrivals within $t = 1/2$ hour is Poisson(λt), and its expectation is $\lambda t = 3/2$.

Problem 2. Two points X_1, X_2 are chosen independently and uniformly on the line segment $[0, a]$, where $a > 0$.

(1) Find the pdf of the random variables $Y = \min(X_1, X_2)$ and $Z = \max(X_1, X_2)$. Make sure to give the answer for all values of the argument of the pdf.

(2) Compute $P\big(X_1 \le a/2,\ X_2 \le a/2\big)$.
(3) Compute $P\big(X_1 \le a/4,\ X_2 \ge 3a/4\big)$ and then the probability that one of the two points, no matter which, lies in $[0, a/4]$ and the remaining point lies in $[3a/4, a]$.
(4) Let I be the indicator of the event "the two points lie in the same half of the segment", i.e. both in $[0, a/2]$ or both in $(a/2, a]$. Compute $E[I]$. Justification required.

Solution: (1) Using independence,

$$(1) P(Y \le y) = P(\min(X_1, X_2) \le y) = 1 - P(X_1 \ge y, X_2 \ge y) = 1 - P(X_1 \ge y)P(X_2 \ge y)$$
$$= 1 - (\frac{a-y}{a})^2 \quad \text{if } 0 \le y \le a.$$

The complete answer is

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ 1 - (\frac{a-y}{a})^2 & 0 \le y \le a \\ 1 & y \ge a \end{cases}$$

and thus,

$$f_Y(y) = \begin{cases} 0 & y \le 0 \text{ or } y \ge a \\ \frac{2}{a^2}(a-y) & 0 \le y \le a \end{cases}$$

Similarly, $P(Z \le z) = P(X_1 \le z)^2 = (\frac{z}{a})^2$ and $f_Z(z) = \frac{2z}{a^2}$ for $0 \le z \le a$ and 0 otherwise.
(2) As above, this probability is $(\frac{1}{2})^2 = \frac{1}{4}$.
(3) The first probability is $\frac{1}{4^2}$ and the second is twice that, namely $1/8$.
(4) $P(I = 0) = 2P(X_1 \le a/2)P(X_2 \ge a/2) = \frac{1}{2}$, so $EI = P(I = 1) = \frac{1}{2}$

Problem 3. (The "Aces in the deck" problem, rephrased) A bag contains 10 balls numbered $1, 2, \ldots, 10$. The balls are drawn one by one uniformly at random *without replacement*. For each of the following questions, you must justify your answer.

(1) Let Y_0 be the number of balls drawn before ball number 1 appears. Find $E[Y_0]$.
(2) Let Y_1 be the number of balls drawn between the times when balls 1 and 2 are drawn (note that ball 1 can be drawn before or after ball 2). Find $E[Y_1]$.
(3) Let Y_2 be the number of balls drawn after both balls 1 and 2 have been drawn. Find $E[Y_2]$.
(4) Find the probability that ball number 1 is drawn before ball number 2.
(5) Let T be the total number of draws made until both balls 1 and 2 have been drawn, so the T^{th} draw is ball 1 or ball 2 depending on which of those two balls appears after the other one. Find $E[T]$.

Solution: We have 2 "labelled" balls in a random permutation of 10 balls.
(1) $P(B_1 = i) = 0.1$ for all $i = 1, \ldots, 10$, so $EY_0 = 0.1(1 + 2 + \cdots + 9) = 0.1 \cdot 45 = 4.5$.
(2)-(3) As with aces in the HW6,P.2, there are 3 gaps "between" the 2 balls, each of which has the same average length equal to 8/3.
(4) The number of permutations in which B_1 is before B_2 is the same as the number when it is after B_2, and each of these permutations has the same probability $1/(10!)$. Thus, the answer is 1/2.
(5) $ET = EY_0 + 1 + EY_1 + 1 = 22/3$, or $ET = 10 - EY_2 = 10 - 8/3 = 22/3$.

Problem 4. Let X be an exponential random variable with parameter $\lambda > 0$, i.e.

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \ge 0.$$

(1) Define $Y = \sqrt{X}$. Find the probability density function (pdf) of Y. Make sure to give the answer for all $y \in (-\infty, \infty)$.

(2) Compute $E[Y]$ in terms of λ (integration by parts and the Gaussian integral may help).
(3) Define $Z = e^{-\lambda X}$. Find the pdf of Z and its expectation (no computations are needed; just recall universality of the uniform).

Solution: (1) The range of Y is from 0 to ∞. For $y \in (0, \infty)$,

$$P(Y \le y) = P(X \le y^2) = \int_0^{y^2} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^{y^2} = 1 - e^{-\lambda y^2}.$$

The RHS is simply $F_X(y^2)$, the CDF of X, computed in class.

$$f_Y(y) = \begin{cases} 0 & -\infty < y < 0 \\ 2y\lambda e^{-\lambda y^2}, & 0 \le y < \infty. \end{cases}$$

Note: this is the change-of-variable formula for the integral $\int_0^t f(x)dx$: we write $x = y^2, dx = 2ydy$ and obtain $\int_0^{\sqrt{t}} f(y^2)2ydy$, so the new pdf is $2yf(y^2)$. We will discuss this later in class.

(2) Integrate $EY = 2\lambda \int_0^\infty y^2 e^{-\lambda y^2} dy$ by parts: let $u = y, dv = ye^{-\lambda y^2}dy$, so $du = dy$, $v = \frac{1}{2\lambda}\int e^{-\lambda y^2} d(\lambda y^2) = -\frac{1}{2\lambda}e^{-\lambda y^2}$, and (multiplying through by 2λ)

$$EY = uv\Big|_0^\infty - \int_0^\infty vdu = -ye^{-\lambda y^2}\Big|_0^\infty + \int_0^\infty e^{-\lambda y^2} dy$$

The first term evaluates to 0, and the second, upon the variable change $y = z/\sqrt{2\lambda}$, becomes a Gaussian, and we obtain

$$EY = \frac{1}{\sqrt{2\lambda}} \int_0^\infty e^{-z^2/2} dz = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}.$$

(3) We know that, for any RV X with CDF $F(x)$, the random variable $F(X) \sim \text{Unif}(0, 1)$. The CDF of X is $1 - e^{-\lambda x}$, and thus $W = 1 - e^{-\lambda X}$ is uniform, but then so is $Z = 1 - W = e^{-\lambda X}$. The pdf of Z is $\mathbb{1}_{(0,1)}$ and $EZ = 1/2$.

Problem 5. Let X be a First Success random variable with parameter p, that is, $P(X = k) = p(1-p)^{k-1}$ for $k = 1, 2, \ldots$.

(1) Define $Y = X - 1$. Find the probability mass function (pmf) of Y and compute $E[Y]$.
(2) Define $Z = \min(X, 3)$. Find the pmf of Z.
(3) Compute $E[Z]$. What happens to $E[Z]$ as $p \to 0$, and why (give an explanation)?

Solution: (1) $Y \sim \text{Geom}(p)$ and thus, $P_Y(k) = p(1-p)^k, k \ge 0$ and $EY = (1-p)/p$.
(2) $Z = \min(X, 3)$. The pmf is

$$P(Z = 1) = P(X = 1) = p,$$
$$P(Z = 2) = P(X = 2) = p(1-p),$$
$$P(Z = 3) = P(X \ge 3) = 1 - P(X = 1) - P(X = 2) = (1-p)^2.$$

(3) The expectation is

$$E[Z] = 1 \cdot p + 2 \cdot p(1-p) + 3 \cdot (1-p)^2 = 3 - 3p + p^2.$$

Intuition: as $p \to 0$, the geometric is very likely to be large, so $\min(X, 3) \to 3$, which matches the formula: $\lim_{p\to 0} E[Z] = 3$.