ENEE324. Midterm examination 2, 4/20/2026. Instructor: A. Barg

- The paper contains 5 problems. Each problem is 10 points. Max score=50 points
- Your answers should be justified. NO CREDIT for an answer with no justification.
- DO NOT copy the problem statement into your paper
- Simplify your calculations as much as you can. Perform divisions, multiplications, cancellations etc.
- SIGN YOUR NAME!

Problem 1. Let X be an exponential random variable with parameter $\lambda > 0$, i.e., the probability density function (pdf)

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \ge 0.$$

(1) Define $Y = \sqrt{X}$. Find the pdf of Y. Specify it for all $y \in (-\infty, \infty)$.
(2) Compute EY in closed form (no integrals left in the answer). Hint: Integrate by parts, then note that $\frac{1}{\sqrt{2\pi\frac{1}{2\lambda}}} e^{-\lambda x^2}$ is the pdf of $\mathcal{N}(0, \frac{1}{2\lambda})$.
(3) Define $Z = \frac{X}{1+X}$. Determine the range of Z.
(4) Compute the probability density function $f_Z(x)$.

Solution: (1) Y is a nonnegative RV with range $[0, \infty)$. Let $x \ge 0$. We have

$$F_Y(x) = P(Y \le x) = P(\sqrt{X} \le x) = P(X \le x^2) = 1 - e^{-\lambda x^2},$$

so $f_Y(x) = 2\lambda x e^{-\lambda x^2}$ for $x \ge 0$ and $= 0$ for $x < 0$,

(2) Integrating by parts with $u = x, dv = 2\lambda x e^{-\lambda x^2} dx = -de^{-\lambda x^2}$, we find

$$EY = \int_0^\infty 2\lambda x^2 e^{-\lambda x^2} dx = -xe^{-\lambda x^2}\Big|_0^\infty + \int_0^\infty e^{-\lambda x^2} dx$$
$$= \sqrt{\frac{\pi}{\lambda}} \left[\frac{1}{\sqrt{2\pi\frac{1}{2\lambda}}} \int_0^\infty e^{-\frac{x^2}{2\cdot\frac{1}{2\lambda}}} dx \right] = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}$$

(3) As t changes from 0 to infinity, $\frac{t}{1+t}$ increases from 0 to 1. Answer: Range$(Z) = [0, 1)$.

(4)

$$P\Big(\frac{X}{1+X} \le x\Big) = P\Big(X \le \frac{x}{1-x}\Big) = 1 - e^{-\lambda\frac{x}{1-x}},$$

so

$$f_Z(x) = \frac{d}{dx}(1 - e^{-\lambda\frac{x}{1-x}}) = \frac{\lambda}{(1-x)^2} e^{-\lambda\frac{x}{1-x}}$$

if $0 \le x < 1$ and 0 for all other x.

Problem 2. Let X be a discrete random variable with values in $\{0, 1, 2, \dots\}$ and finite variance. Define

$$g(X) = \mathbf{1}_{\{X \text{ is even}\}},$$

i.e., $g(X)$ is the indicator of the event $\{X \text{ is even}\}$.

(1) Show that

$$\mathrm{Var}(g(X)) = E[g(X)]\big(1 - E[g(X)]\big).$$

(2) Suppose $X \sim \mathrm{FS}(p)$ (First Success) with

$$P(X = k) = p(1-p)^{k-1}, \qquad k = 1, 2, \dots.$$

Compute $E[g(X)]$ in closed form (no sums left in the answer).

(3) Deduce a closed-form expression for $\mathrm{Var}(g(X))$.

(4) Evaluate $\mathrm{Var}(g(X))$ in the cases $p \to 0$ and $p \to 1$, and briefly interpret the result.

Solution: (1) $E(g(X)^2) = E(g(X)) = P(X \text{ even})$, so

$$\text{Var}\,(g(X)) = E(g(X)) - (E(g(X))^2 = Eg(X)(1 - Eg(X)).$$

(2)

$$E[g(X)] = P(X \text{ even}) = \sum_{k=1}^{\infty} p(1-p)^{2k-1} = p\frac{1-p}{1-(1-p)^2} = \frac{1-p}{2-p}.$$

(3)

$$\text{Var}\,(X) = Eg(X)(1 - Eg(X)) = \frac{1-p}{2-p}\left(1 - \frac{1-p}{2-p}\right) = \frac{1-p}{(2-p)^2}.$$

(4) For $p = 0$ we obtain $\text{Var}\,(g(X)) = \frac{1}{4}$ and for $p \to 1$, $\text{Var}\,(g(X)) \to 0$ since success is virtually guaranteed on the first try, and the random variable X is almost a constant, equal to 1, so $g(X) = 0$.

Problem 3. A city has n parking spots and m drivers, where $m \geq n$. Each driver independently chooses one of the n spots uniformly at random and parks there if the spot is empty; otherwise the driver leaves.

Let X be the number of occupied parking spots after all m drivers have arrived and attempted to park.

(1) For $i = 1, \ldots, n$, define the indicator random variable I_i for the event that spot i is occupied. Express X in terms of $I_1, \ldots, I_n$.

(2) Compute $E[I_i]$.

(3) Use linearity of expectation to compute $E[X]$.

(4) Give a simple asymptotic approximation for $E[X]$ when n is large and $m = cn$ for a fixed constant $c > 0$.

Solution: (1) Let I_i be the indicator the the ith spot is occupied. Then we have $X = \sum_{i=1}^{n} I_i$.

(2) The probability that a spot is vacant equals $(1 - \frac{1}{n})^m$, so

$$EI_i = 1 - \Big(1 - \frac{1}{n}\Big)^m.$$

(3) We obtain

$$EX = n\Big(1 - \Big(1 - \frac{1}{n}\Big)^m\Big)$$

(4) We have

$$EX = n\Big(1 - \Big(1 - \frac{1}{n}\Big)^{cn}\Big) \approx n(1 - e^{-c}).$$

Problem 4. The number of emails a server receives in one hour is a random variable $X \sim \text{Poisson}(\lambda)$.

(1) Compute $P(X = 0)$ and $P(X = 1)$.

(2) Suppose the hour is divided into two half-hour intervals. Let X_1 and X_2 denote the numbers of emails received in the first and second half-hour, respectively. Assume that X_1 and X_2 are independent and each has distribution $\text{Poisson}(\lambda/2)$. Compute $P(X_1 = 1,\ X_2 = 0)$.

(3) Show that
$$P(X = 1) = P(X_1 = 1, X_2 = 0) + P(X_1 = 0, X_2 = 1).$$

(4) Generalize part (3): for an integer $k \geq 0$, express $P(X = k)$ in terms of X_1 and X_2.

(5) Using the binomial theorem, verify that $X_1 + X_2 \sim \text{Poisson}(\lambda)$.

Solution: (1) Since $P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$, we have $P(X = 0) = e^{-\lambda}$ and $P(X = 1) = \lambda e^{-\lambda}$.

(2) By independence,
$$P(X_1 = 1, X_2 = 0) = e^{-\frac{\lambda}{2}}\frac{\lambda}{2}e^{-\frac{\lambda}{2}} = \frac{\lambda}{2}e^{-\lambda}.$$

(3) The event $\{X = 1\}$ is a disjoint union of the events $\{X_1 = 1, X_2 = 0\}$ and $\{X_1 = 0, X_2 = 1\}$, so the formula follows by the additivity axiom. In terms of the Poisson distributions, $P(X = 1) = \lambda e^{-\lambda}$ and
$$P(X_1 = 1, X_2 = 0) = \frac{\lambda}{2}e^{-\lambda} = P(X_1 = 0, X_2 = 1),$$
so they add up to $P(X = 1)$.

(4)-(5)
$$P(X = k) = \sum_{i=0}^{k} P(X_1 = i, X_2 = k - i) = e^{-\lambda}\sum_{i=0}^{k}\left(\frac{\lambda}{2}\right)^i \frac{1}{i!}\left(\frac{\lambda}{2}\right)^{k-i}\frac{1}{(k-i)!}$$
$$= \frac{e^{-\lambda}\lambda^k}{k!2^k}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!} = \frac{e^{-\lambda}\lambda^k}{k!}.$$

In the last step we used $\sum_{i=0}^{k}\binom{k}{i} = 2^k$ (the binomial theorem). Thus, $X \sim \text{Poisson}\,(\lambda)$, as required.

Problem 5. A group of n people independently choose one of m restaurants uniformly at random for dinner.

(1) Fix a number k, $1 \le k \le n$. Find the expected number of restaurants that are chosen by exactly k people.

(2) Find the expected number of restaurants that are chosen by at least 2 people.

(3) Let Y be the number of pairs of people who choose the same restaurant. Compute EY.

Solution: (1) Probability that k people choose the same restaurant equals $(\frac{1}{m})^k(\frac{m-1}{m})^{n-k}$. Since there are $\binom{n}{k}$ ways to choose a group of k people, the probability that restaurant $i, 1 \le i \le m$ is chosen equals

$$p = \binom{n}{k}\left(\frac{1}{m}\right)^k\left(\frac{m-1}{m}\right)^{n-k}.$$

Let X be the random number of restaurants chosen by exactly k people. By linearity of expectation,

$$EX = mp = m\binom{n}{k}\left(\frac{1}{m}\right)^k\left(\frac{m-1}{m}\right)^{n-k}.$$

(2) Denote by Y the number of restaurants chosen by ≥ 2 people. Let R_i be the event that restaurant i is chosen by ≥ 2 people, then

$$P(R_i) = 1 - \left(1 - \frac{1}{m}\right)^n - \frac{n}{m}\left(1 - \frac{1}{m}\right)^{n-1}.$$

Using indicators of the events R_i and linearity of expectation, we obtain

$$EY = m\Big[1 - \left(1 - \frac{1}{m}\right)^n - \frac{n}{m}\left(1 - \frac{1}{m}\right)^{n-1}\Big]$$

(3) The probability that a given pair chooses the same restaurant is $1/m^2$. The expected number of pairs is $\frac{n(n-1)}{2m^2}$.