ENEE324, Midterm Exam 2. November 12, 2026 Instructor: Alexander Barg

- Closed book; one two-sided sheet of handwritten notes is permitted. Calculators are permitted.
- Justification is required: a numerical answer with no reasoning receives no credit.
- Exact answers (fractions, powers, e^{-a}, values of Φ) are acceptable everywhere; you need not evaluate them as decimals.
- **Five problems**, each problem is worth 20 points, for a total of 100.

Problem 1. A fair coin is tossed 10 times. Let X be the number of times that two consecutive tosses are both heads; for example, the sequence HHHTTHHTTT has $X = 3$ (positions 1–2, 2–3, and 6–7).

(a) Write X as a sum of indicator random variables and compute $\mathbb{E}[X]$.

(b) For two of your indicators I and J, compute $\mathbb{E}[IJ]$ when they refer to overlapping pairs of tosses (such as positions 1–2 and 2–3), and when they refer to disjoint pairs.

(c) Compute $\mathrm{Var}(X)$. Hint: To compute EX^2, write X as the sum in part (a), square it and use the results of part (b) to compute the terms in the resulting expression.

Solution. *(a) For $i = 1, \dots, 9$ let I_i be the indicator that tosses i and $i+1$ are both heads. Then $X = I_1 + \cdots + I_9$ and $\mathbb{E}[I_i] = P(HH) = \frac{1}{4}$, so by linearity $\mathbb{E}[X] = 9 \cdot \frac{1}{4} = \frac{9}{4}$.*

(b) $I_i I_{i+1} = 1$ exactly when tosses $i, i+1, i+2$ are all heads, so $\mathbb{E}[I_i I_{i+1}] = \frac{1}{8}$. If $|i-j| \geq 2$, the two pairs use four different tosses, which are independent, so $\mathbb{E}[I_i I_j] = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

(c) $\mathbb{E}[X^2] = \sum_i \mathbb{E}[I_i^2] + \sum_{i \neq j} \mathbb{E}[I_i I_j]$. Since $I_i^2 = I_i$, the first sum is $\frac{9}{4}$. Among the $9 \cdot 8 = 72$ ordered pairs (i, j) with $i \neq j$, there are $2 \cdot 8 = 16$ adjacent ones and 56 others. Hence

$$\mathbb{E}[X^2] = \frac{9}{4} + 16 \cdot \frac{1}{8} + 56 \cdot \frac{1}{16} = \frac{9}{4} + 2 + \frac{7}{2} = \frac{31}{4}, \qquad \mathrm{Var}(X) = \frac{31}{4} - \frac{81}{16} = \frac{43}{16} \approx 2.69.$$

(For comparison, 9 independent indicators with mean $\frac{1}{4}$ would give $9 \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{27}{16}$; the overlapping pairs are positively correlated, which raises the variance.)

Problem 2. A batch contains 500 chips; each is defective with probability 0.004, independently of the others. Let X be the number of defective chips in the batch.

(a) Name the exact distribution of X, and compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

(b) Use the Poisson approximation to estimate $P(X \leq 1)$.

(c) Use Chebyshev's inequality to bound $P(X \geq 5)$. (Hint: $X \geq 5$ means $X - \mathbb{E}[X] \geq 3$.)

(d) A second, independent batch of 500 chips is produced in the same way, and the two batches together contain exactly 3 defective chips. Using the Poisson approximation for each batch, find the probability that all 3 of them are in the first batch. Hint: Compute $P(X = 3 \mid X+Y = 3)$ using the definition of conditional probability.

Solution. *(a) $X \sim \mathrm{Bin}(500, 0.004)$, so $\mathbb{E}[X] = 500 \cdot 0.004 = 2$ and $\mathrm{Var}(X) = 500 \cdot 0.004 \cdot 0.996 = 1.992$.*

(b) $X \approx \mathrm{Pois}(\lambda)$ with $\lambda = np = 2$, so $P(X \leq 1) \approx e^{-2}(1+2) = 3e^{-2} \approx 0.406$. (The exact binomial value is also 0.405 to three decimals.)

(c) Since $X \geq 5$ implies $|X - 2| \geq 3$,

$$P(X \geq 5) \leq P(|X-2| \geq 3) \leq \frac{\mathrm{Var}(X)}{3^2} = \frac{1.992}{9} \approx 0.22.$$

(The actual value is about 0.053; Chebyshev uses only the mean and the variance.)

(d) Let $X \approx \mathrm{Pois}(2)$ and $Y \approx \mathrm{Pois}(2)$ be the counts in the two batches, independent; then $X+Y \approx \mathrm{Pois}(4)$, and

$$P(X=3 \mid X+Y=3) = \frac{P(X=3)\,P(Y=0)}{P(X+Y=3)} = \frac{\frac{2^3}{3!}e^{-2}\cdot e^{-2}}{\frac{4^3}{3!}e^{-4}} = \Big(\frac{1}{2}\Big)^3 = \frac{1}{8}.$$

Equivalently, given $X+Y=3$, $X \sim \mathrm{Bin}\big(3, \frac{2}{2+2}\big)$.

Problem 3. A random variable X has density

$$f(x) = c\,x \quad \text{for } 0<x<2, \qquad f(x)=0 \quad \text{otherwise.}$$

(a) Find c and the CDF $F(x)$ for all real x.

(b) Compute $\mathbb{E}[X]$, $\mathbb{E}[X^2]$, and $\mathrm{Var}(X)$.

(c) Find the median of X, and $P(X > \mathbb{E}[X])$.

(d) Find the CDF of $Y = X^2$ and identify the distribution of Y. Check your answer against $\mathbb{E}[X^2]$ from part (b).

Solution. *(a) $\int_0^2 cx\,dx = 2c = 1$, so $c = \frac{1}{2}$. Then*

$$F(x) = \begin{cases} 0, & x<0,\\ x^2/4, & 0\le x<2,\\ 1, & x\ge 2. \end{cases}$$

(b) $\mathbb{E}[X] = \int_0^2 x\cdot\frac{x}{2}\,dx = \frac{4}{3}$; by LOTUS $\mathbb{E}[X^2] = \int_0^2 x^2\cdot\frac{x}{2}\,dx = 2$; hence $\mathrm{Var}(X) = 2 - \frac{16}{9} = \frac{2}{9}$.

(c) $m^2/4 = \frac{1}{2}$ gives $m = \sqrt{2} \approx 1.414$. Further, $P(X > \frac{4}{3}) = 1 - \frac{(4/3)^2}{4} = 1 - \frac{4}{9} = \frac{5}{9}$.

(d) Y takes values in $(0,4)$. For $0 \le y \le 4$, since $X>0$,

$$F_Y(y) = P(X^2 \le y) = P(X \le \sqrt{y}) = F(\sqrt{y}) = \frac{y}{4},$$

with $F_Y(y) = 0$ for $y<0$ and $F_Y(y) = 1$ for $y>4$. Thus $Y \sim \mathrm{Unif}(0,4)$, and $\mathbb{E}[Y] = 2 = \mathbb{E}[X^2]$, as it should be.

Problem 4. The output voltage of a regulator is $V \sim \mathcal{N}(5,\ 0.01)$ volts (Gaussian with mean 5, variance 0.01).

(a) Compute $P(4.85 < V < 5.2)$.

(b) Find the voltage v that is exceeded by only 1% of the regulators.

(c) A regulator is out of specification if $|V-5| > 0.2$. Find the probability q that a regulator is out of specification.

(d) Ten regulators, with independent outputs, are installed in a system. Find the probability that at least one of them is out of specification.

You may use: $\Phi(1.5) = 0.9332$, $\Phi(2) = 0.9772$, $\Phi^{-1}(0.99) = 2.326$, where Φ is the $\mathcal{N}(0,1)$ CDF.

Solution. *Write $Z = (V-5)/0.1 \sim \mathcal{N}(0,1)$, so $P(V \le x) = \Phi\big(\frac{x-5}{0.1}\big)$, and use $\Phi(-z) = 1-\Phi(z)$.*

(a) $P(4.85 < V < 5.2) = \Phi(2) - \Phi(-1.5) = \Phi(2) + \Phi(1.5) - 1 = 0.9772 + 0.9332 - 1 = 0.9104$.

(b) $P(V > v) = 0.01$ means $\frac{v-5}{0.1} = \Phi^{-1}(0.99) = 2.326$, so $v \approx 5.233$ V.

(c) $q = P(|Z| > 2) = 2\big(1-\Phi(2)\big) = 0.0456$.

(d) The number of out-of-specification regulators is $\mathrm{Bin}(10, q)$, so

$$P(\text{at least one}) = 1 - (1-q)^{10} = 1 - 0.9544^{10} \approx 0.373.$$

(e) $W = 2V - 3 = 2(5 + 0.1Z) - 3 = 7 + 0.2Z$*, so* $W \sim \mathcal{N}(7,\, 0.04)$*: mean* 7*, standard deviation* 0.2*.*

Problem 5. Requests arrive at a server as a Poisson process with rate 3 per minute. Let N_t be the number of requests in the first t minutes.

(a) Find the probability that no request arrives during a given 30-second interval.

(b) Find the probability that **exactly** 2 requests arrive during the first minute and **at least** one request arrives during the second minute.

(c) Let T_4 be the arrival time of the 4th request. Find $\mathbb{E}[T_4]$ and $\mathrm{Var}(T_4)$. Hint: you do not need the distribution of T_4: use the fact that inter-arrival times are independent.

(d) Requests are of two types: type A arrive as a Poisson process with rate 2 per minute, and type B, independently, with rate 1 per minute (together they form the stream above). What is the probability that the next request to arrive is of type B? Hint: Use the fact that $P(T_B < T_A) = \int_0^\infty P(T_A > x) f_B(x) dx$.

Solution. *(a)* $N_{1/2} \sim \mathrm{Pois}(1.5)$*, so* $P(N_{1/2} = 0) = e^{-1.5} \approx 0.223$*.*

(b) The counts in the first and second minutes are independent $\mathrm{Pois}(3)$*:*

$$P(N_1 = 2)\, P(N_2 - N_1 \geq 1) = \frac{3^2}{2!} e^{-3} \left(1 - e^{-3}\right) = \frac{9}{2} e^{-3} \left(1 - e^{-3}\right) \approx 0.213.$$

(c) T_4 *is the sum of* 4 *independent* $\mathrm{Exp}(3)$ *inter-arrival times, each with mean* $\frac{1}{3}$ *and variance* $\frac{1}{9}$*. Hence* $\mathbb{E}[T_4] = \frac{4}{3}$ *minutes and* $\mathrm{Var}(T_4) = \frac{4}{9}$ *minutes*^{2}*.*

(d) The waiting times for the next type-A and the next type-B request are independent, $\mathrm{Exp}(2)$ *and* $\mathrm{Exp}(1)$*. Type B comes first with probability*

$$P(T_B < T_A) = \int_0^\infty (1 - e^{-t})\, 2e^{-2t}\, dt = 1 - \frac{2}{3} = \frac{1}{3}.$$

Compare with the general answer for $P(X_i$ *is the smallest*$)$ *in Lec. 17, p.4.*