ENEE 324. Home assignment 3, solutions

1. $p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k=0,1,\ldots$

$\mathbb{E}[X] = \lambda \Rightarrow \lambda = 3$

$\mathbb{P}[X \le 3] = e^{-3}\left(\sum_{k=0}^{3} \frac{3^k}{k!}\right) = e^{-3}\left(1+3+\frac{9}{2}+\frac{9}{2}\right) \cong 0.6472$

2. Electron - 0.2 Proton - 0.3 Neutrino - 0.4 No Particle - 0.1

(a) P(1st electron in 4^{th} observation) $= (1-0.2)^3 \cdot 0.2 = 0.0512$

(b) P(1st proton no earlier than 5^{th} observation)

$=$ P(First 4 observations are not protons) $= (1-0.3)^4 = 0.2401$

(c) P(10 observations have 3E, 2P, 1N, 4 empty slots)

$= \binom{10}{3,2,1,4}(0.2)^3(0.3)^2(0.4)(0.1)^4$

3. (We are assuming that X and Y are independent)

(a) $$P(X=i, Y=10-i) = p(1-p)^{i-1} p (1-p)^{10-i-1} = p^2(1-p)^8, \quad i=1,2,\ldots,9$$

$$p_{X+Y}(10) = \sum_{i=1}^{9} P(X=i, Y=10-i) = \sum_{i=1}^{9} p^2(1-p)^8 = 9p^2(1-p)^8$$

(b) $$p_{X|X+Y}(k|10) = \frac{P(X=k \cap X+Y=10)}{P(X+Y=10)}$$

$$= \frac{p_X(k)\, p_Y(10-k)}{9p^2(1-p)^8} = \frac{p^2(1-p)^8}{9p^2(1-p)^8} = \frac{1}{9} \quad \text{if } k=1,2,\ldots,9$$

$$p_{X|X+Y}(k|10) = 0 \text{ otherwise}$$

4. $X_1 \sim \text{Bernoulli}(p_1)$, $X_2 \sim \text{Bernoulli}(p_2)$

$$Z = X_1 - X_2$$

$$p_Z(i) = \begin{cases} P(X_1=0, X_2=1) = (1-p_1)p_2, & i=-1 \\ P(X_1=0, X_2=0) + P(X_1=1, X_2=1) = (1-p_1)(1-p_2) + p_1p_2, & i=0 \\ P(X_1=1, X_2=0) = p_1(1-p_2), & i=1 \end{cases}$$

$$E[Z] = E[X_1] - E[X_2] = p_1 - p_2$$

$$E[Z^2] = 1 \cdot ((1-p_1)p_2 + p_1(1-p_2)) + 0 = p_1 + p_2 - 2p_1p_2$$

$$\text{var}(Z) = E[Z^2] - (E[Z])^2 = p_1 + p_2 - 2p_1p_2 - p_1^2 + 2p_1p_2 - p_2^2$$

$$= p_1(1-p_1) + p_2(1-p_2)$$

5. Range $(Z) = \{0,1,2,3\}$

$$p_Z(0) = P(X=0 \cap Y \in \{1,3,4\}) = 0.2 \cdot 1 = 0.2$$

$$p_Z(1) = P(X=1 \cap Y \in \{1,3,4\}) + P(X \in \{2,3\} \cap Y=1)$$

$$= 0.3 \cdot 1 + (0.4+0.1) \cdot 0.7 = 0.3 + 0.35 = 0.65$$

$$p_Z(2) = P(X=2 \cap Y \in \{3,4\}) = 0.4 \times (0.2+0.1) = 0.12$$

$$p_Z(3) = P(X=3 \cap Y \in \{3,4\}) = 0.1 \times (0.2+0.1) = 0.03$$

$$p_Z(i) = 0, \quad i \neq 0,1,2,3$$

6. $$p_Y(0) = \sum_{x=0}^{1} p_{X,Y}(x,0) = \tfrac{1}{2}(1-p) + \tfrac{1}{2}p = \tfrac{1}{2}$$

$$p_Y(1) = \sum_{x=0}^{1} p_{X,Y}(x,1) = \tfrac{1}{2}p + \tfrac{1}{2}(1-p) = \tfrac{1}{2} = 1 - p_Y(0)$$

$$p_{Y|X}(0|1) = \frac{p_{XY}(1,0)}{p_X(1)} = \frac{\frac{1}{2}p}{p_{X,Y}(1,0) + p_{X,Y}(1,1)} = \frac{\frac{1}{2}p}{\frac{1}{2}p + \frac{1}{2}(1-p)} = p$$

$$p_{Y|X}(0|0) = \frac{p_{XY}(0,0)}{p_X(0)} = \frac{\frac{1}{2}(1-p)}{1-p_X(1)} = \frac{\frac{1}{2}(1-p)}{\frac{1}{2}} = 1-p$$

Thus, Y and X are connected via a Binary Symmetric Channel (lect. 6)

ENEE324, Solutions of HW4

Prob.1

(a) $a\int_0^{\infty} e^{-0.02x}\,dx = -\frac{a}{0.02}e^{-0.02x}\Big|_0^{\infty} = \frac{a}{0.02} = 1 \Rightarrow a = 0.02$

$\cancel{(b)}$ $F_Y(y) = P(Y \le y) = P(2X+12 < y) = P\left(X < \frac{1}{2}(y-12)\right) = \boxed{F_X\left(\frac{1}{2}(y-12)\right)}\ \boxed{(d)}$

$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = 0.02\int_{-\infty}^{x} e^{-0.02t}\,dt = -e^{-0.02t}\Big|_{0}^{x} = 1-e^{-0.02x}$

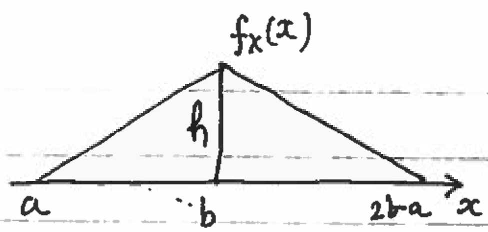
$F_Y(y) = F_X\left(\frac{1}{2}(y-12)\right) = 1-e^{-0.01(y-12)} = 1-\frac{e^{-0.01y}}{e^{0.12}} \cong 1-0.887\,e^{-0.01y}$ $\leftarrow$ ans. for (d)

(c) $P(Y<50) = F_Y(50) \approx 0.462$

(b) $EY = 2EX + 12 = 2\cdot\frac{1}{0.02} + 12 = 112$ //

Prob 2.

(a)



To find h, use normalization $1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \frac{1}{2}h\left[(b-a)+(b-a)\right] = h(b-a)$

$$\boxed{h = \frac{1}{b-a}}$$

(b) $EX = \int_a^b \frac{x-a}{(b-a)^2}x\,dx + \int_b^{2b-a} -\frac{x+a-2b}{(a-b)^2}x\,dx = \frac{1}{(b-a)^2}\left(\frac{x^3}{3}-\frac{ax^2}{2}\right)\Big|_a^b$

$-\frac{1}{(b-a)^2}\left(\frac{x^3}{3}+\frac{x^2}{2}(a-2b)\right)\Big|_b^{2b-a} = b$

$Var(X) = EX^2 - E(X)^2 = \frac{(a-b)^2}{6}$ (on paper or by computer)

(c) $P\left(|X-b| \le \frac{a-b}{\sqrt{6}}\right) = \int_{b-\frac{a-b}{\sqrt{6}}}^{b} \frac{x-a}{(b-a)^2}\,dx + \int_b^{b+\frac{a-b}{\sqrt{6}}} \frac{2b-x-a}{(b-a)^2}\,dx = \frac{2\sqrt{6}-1}{6}$

Prob 3.

(a)

$$P_{Y|X}(j|i) = \begin{array}{cccc|c} & & & & Y \\ 0 & 0 & 0 & 1/4 & 4 \\ 0 & 0 & 1/3 & 1/4 & 3 \\ 0 & 1/2 & 1/3 & 1/4 & 2 \\ 1 & 1/2 & 1/3 & 1/4 & 1 \\ \hline 1 & 2 & 3 & 4 & x \end{array}$$

$$P_{XY}(i,j) = \begin{array}{cccc|c} & & & & Y \\ 0 & 0 & 0 & 1/16 & 4 \\ 0 & 0 & 1/12 & 1/16 & 3 \\ 0 & 1/8 & 1/12 & 1/16 & 2 \\ 1/4 & 1/8 & 1/12 & 1/16 & 1 \\ \hline 1 & 2 & 3 & 4 & \leftarrow X \end{array}$$

(b)

$$P_Y(i) = \begin{array}{c|cccc} i & 1 & 2 & 3 & 4 \\ \hline & \frac{25}{48} & \frac{13}{48} & \frac{7}{48} & \frac{1}{16} \end{array} \qquad P(2 \le Y \le 3) = \frac{20}{48} = \frac{5}{12}$$

(c)

$$P_{X|Y}(i|2) = \frac{P_{XY}(i,2)}{P_Y(2)} = \begin{array}{c|c|c|c|} i \; 1 & 2 & 3 & 4 \\ \hline 0 & \frac{6}{13} & \frac{4}{13} & \frac{3}{13} \end{array}$$

(d)

$$E[X \mid \{2 \le Y \le 3\}] = \sum_{i=1}^{4} i \cdot P_{X|A}(i|A)$$

$$P_{X|A}(i) = \frac{P(X=i \cap 2 \le Y \le 3)}{\underbrace{P(2 \le Y \le 3)}_{= \cancel{10/13}\ 5/12}} = \frac{12}{5} \times \begin{Bmatrix} i=1 & i=2 & i=3 & i=4 \\ 0 & \frac{1}{8} & \frac{1}{6} & \frac{1}{8} \end{Bmatrix}$$

$$= \begin{array}{c|c|c|c|} i=1 & 2 & 3 & 4 \\ \hline 0 & \frac{3}{10} & \frac{4}{10} & \frac{3}{10} \end{array}$$

Prob. 5

$$P(X \le 11) = F_X(11) = \Phi\left(\frac{11-12}{\sqrt{10}}\right) = \Phi(-0.316) = 1 - \Phi(0.316)$$

$$\cong 1 - 0.62 \approx 0.38 \quad (\text{more accurately, } 0.3759)$$

Since $\mathrm{erf}(x) = 2\Phi(\sqrt{2}\,x) - 1$

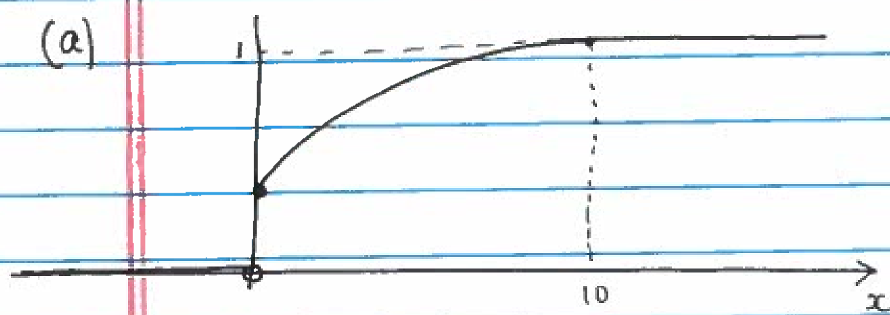
$$\Phi(z) = \frac{1}{2}\left(\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right) + 1\right), \quad z \ge 0$$

$$P(X \le 11) = 1 - \Phi(0.316) = \frac{1}{2}\left(1 - \mathrm{erf}(0.223)\right)$$

4

$$F_X(x) = \begin{cases} 0 & x > 0 \\ 0.5 + 0.1x - 0.005x^2 & 0 \le x \le 10 \\ 1 & x \geq 10 \end{cases}$$

(a)



(b)

$$f_X(x) = \begin{cases} \frac{1}{2}\delta(x) + 0.1 - 0.01x & 0 \le x \le 10 \\ 0 & x \geq 10 \end{cases}$$

(c)

$$EK = E\left[\frac{10X^2}{2}\right] = 5\,EX^2 = 5\int_0^{10}\left(\frac{x^2}{2}\delta(x) + 0.1x^2 - 0.01x^3\right)dx$$

$$= 5\left(0.1\frac{x^3}{3} - 0.01\frac{x^4}{4}\right)\Big|_{x=0}^{10} = 5\left(\frac{100}{3} - \frac{100}{4}\right) = \frac{500}{12} = \frac{125}{6}$$

(d)

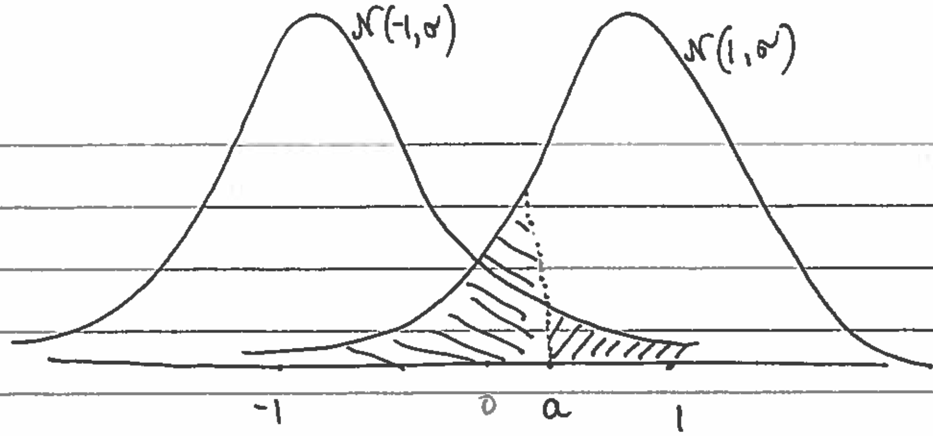
$$F_K(x) = P(K < x) = P\left(\tfrac{1}{2}mX^2 < x\right) = P\left(X^2 < \frac{2x}{m}\right) = P\left(X < \sqrt{\tfrac{x}{5}}\right) = F_X\left(\sqrt{\tfrac{x}{5}}\right)$$

$$F_K(x) = \begin{cases} 0 & x < 0 \\ 0.5 + 0.1\sqrt{\frac{x}{5}} - 0.001x & 0 \le x \le \frac{10\cdot(10)^2}{2} = 500 \\ 1 & x \geq 500 \end{cases}$$

$$f_K(x) = 0.5\,\delta(x) + \frac{0.05}{\sqrt{5x}} - 0.001 \quad \text{if } 0 \le x \le 500$$

and $f_K(x) = 0$ o/w

Prob. 6



(a) Let $E = \{\text{incorrect decision}\}$ be the event of interest

$$P(E) = P(E|+1)p + P(E|-1)(1-p)$$

$$= (1-p)\cdot[\text{the area of } \text{ region}] + \cancel{1-p}\; p\cdot[\text{the area of } \text{ reg.}]$$

$$= (1-p)\,(1 - F_{-1}(a)) + p\,(F_1(a))$$

where $F_{-1}(z) =$ CDF of $\mathcal{N}(-1,\sigma)$

$F_1(z) =$ CDF of $\mathcal{N}(1,\sigma)$

$$= (1-p)\left(1 - \Phi\left(\frac{a+1}{\sigma}\right)\right) + p\,\Phi\left(\frac{a-1}{\sigma}\right)$$

(b) Substitute $p = \frac{1}{3}$, $\sigma = \frac{1}{\sqrt{2}}$, $a = 0.1$:

$$P(E) = \frac{2}{3}\left(1 - \Phi(1.1\sqrt{2})\right) + \frac{1}{3}\Phi\left(-0.9\sqrt{2}\right) \approx 0.0738$$

Problem 7 : Random resistance $X \sim \mathcal{N}(1000, \sigma^2)$; denote CDF by F_X

We are given that $P(|X - 1000| \leq 100) = 0.8$

$$F_X(1100) - F_X(900) = \Phi\left(\frac{100}{\sigma}\right) - \Phi\left(-\frac{100}{\sigma}\right) = 2\Phi\left(\frac{100}{\sigma}\right) - 1 = 0.8$$

By computer, $\sigma \approx 78$ (more accurately, 78.03)

Now let $Y \sim \mathcal{N}(1050, \sigma^2)$; find

$$P(900 \leq Y \leq 1100) = F_Y(1100) - F_Y(900) = \Phi\left(\frac{50}{\sigma}\right) - \Phi\left(-\frac{150}{\sigma}\right) \approx 0.712$$

Answer : The new yield $\approx 71.2\%$

ENEE 324. Home assignment 5, Solutions

1. Let Y be the error of the reading, $X=|Y|$ be its absolute value. We know that $Y \sim \mathcal{N}(0,\sigma^2)$ for some unknown σ.

Compute $F_X(x)$: $\quad F_X(x)=0$ if $x<0$

$$F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(-x \le Y \le x) = 2\Phi\left(\frac{x}{\sigma}\right)-1, \quad x \geq 0$$

$$f_X(x) = \frac{d}{dx} F_X(x) = 2 \cdot \frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} e^{-x^2/2\sigma^2}, \quad x \geq 0$$

$$= 0 \qquad x<0.$$

$$\mathbb{E}[X] = \int_0^\infty x f_X(x)\,dx = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \int_0^\infty x e^{-x^2/2\sigma^2} dx = \sqrt{\frac{2}{\pi}}\,\sigma.$$

We need to find

$$\mathbb{P}(Y > \mathbb{E}[X]) = 1 - \Phi\left(\frac{\mathbb{E}[X]}{\sigma}\right) = 1-\Phi\left(\sqrt{\frac{2}{\pi}}\right) = 1-\Phi(0.7979)$$

$$\approx 0.212$$

(observe that the answer does not depend on the actual value of $\mathbb{E}[X]=0.25$)

Problem 2. Let X_1, X_2, X_3, X_4 be the RVs for the finish time of the runners. We have for the team race time Z:

$$Z = X_1 + X_2 + X_3 + X_4 \sim \mathcal{N}\left(4 \cdot 52,\ 4 \cdot (0.8)^2\right) = \mathcal{N}\left(208, (1.6)^2\right)$$

$$P(Z \leq 206) = F_Z(206) = \Phi\left(\frac{206-\mu_Z}{\sigma_Z}\right) = \Phi(-1.25) = 1 - \Phi(1.25) \cong 0.1056$$

(note that by the problem statement $P(X_i < 0)$ is positive, which makes no physical sense, but is practically OK to ignore).

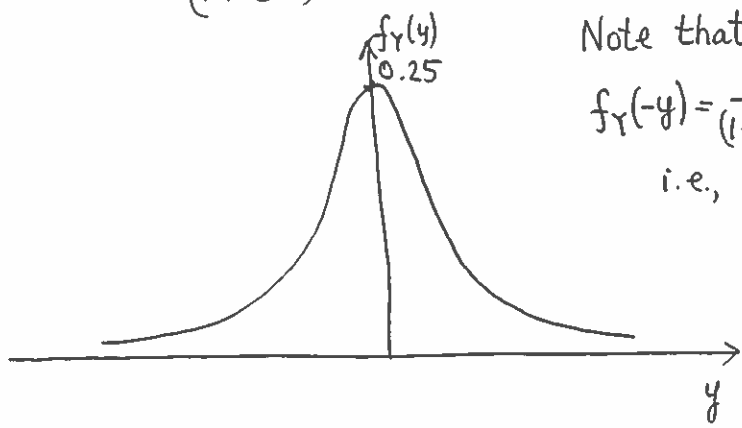
Problem 3: see next page.

Problem 4: We have
$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 1 & x > 1. \end{cases}$$

$$F_Y(y) = P(Y \leq y) = P\left(\ln \frac{X}{1-X} < y\right) = P\left(X \leq \frac{1}{1+e^{-y}}\right)$$

$$= F_X\left(\frac{1}{1+e^{-y}}\right) = \frac{1}{1+e^{-y}}, \quad -\infty < y < \infty \quad \left(\text{note that } 0 < \frac{1}{1+e^{-y}} < 1 \text{ for all } y\right)$$

$$f_Y(y) = F_Y'(y) = \frac{e^{-y}}{(1+e^{-y})^2}, \quad -\infty < y < \infty$$



Note that
$$f_Y(-y) = \frac{e^{y}}{(1+e^{y})^2} = \frac{e^{-y}}{(1+e^{-y})^2} = f_Y(y)$$
i.e., $f_Y(y)$ is an even function

3. Find c from $\iint f_{X,Y}(x,y)\,dx\,dy = 1$

$$\int_0^\infty\int_0^x e^{-x-y}\,dx\,dy = \int_0^\infty dx\,\left(-e^{-x-y}\right)\Big|_0^x = \int_0^\infty dx\left(-e^{-2x}+e^{-x}\right) = \frac{1}{2}e^{-2x}\Big|_0^\infty - e^{-x}\Big|_0^\infty$$

$$= 1-\frac{1}{2} = \frac{1}{2} \quad \Rightarrow \quad c = 2$$

$$f_X(x) = \int f_{X,Y}(x,y)\,dy = \int_0^x 2e^{-x-y}\,dy = 2\left(-e^{-x-y}\right)\Big|_0^x = 2\left(e^{-x}-e^{-2x}\right),\ x \geq 0$$

$$f_Y(y) = \int_y^\infty 2e^{-x-y}\,dx = 2\left(-e^{-x-y}\right)\Big|_y^\infty = 2e^{-2y},\ y \geq 0$$

$$F_{X,Y}(x,y) = \int_0^y\int_y^x 2e^{-x-y}\,dx\,dy = 1-e^{-2y} - 2\left(e^{-x}-e^{x-y}\right) \qquad 0 \leq y \leq x < \infty$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2e^{-x-y}}{2e^{-2y}} = e^{-x+y}$$

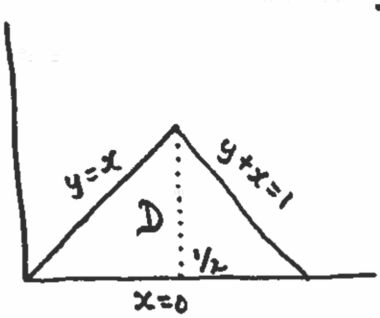
$$\mathbb{E}\left[X^2|Y=2\right] = \int_{2}^\infty x^2 e^{-x+2}\,dx = -e^{-x+2}x^2\Big|_2^\infty + \int 2x e^{-x+2}\,dx$$

$2 \uparrow$ since $y \leq x$

$$= 4 - 2e^{-x+2}x\Big|_2^\infty + 2\int e^{-x+2}\,dx$$

$$= 4+4+2 = 10 \,//$$

$$P\left[X+Y\leq 1\right] = 2\iint_D e^{-x-y}\,dx\,dy = 2\int_{x=0}^{1/2}\int_{y=0}^{x} e^{-x-y}\,dx\,dy + 2\int_{x=1/2}^{1}\int_{y=0}^{1-x} e^{-x-y}\,dx\,dy$$



$$= 2\int_0^{1/2} -e^{-x-y}\Big|_{y=0}^{x}\,dx + 2\int_{1/2}^{1} -e^{-x-y}\Big|_0^{1-x}\,dx$$

$$= 2\int_0^{1/2}\left(e^{-x}-e^{-2x}\right)dx + 2\int_{1/2}^{1}\left(e^{-x}-\frac{1}{e}\right)dx$$

$$= 2\left(1-\frac{1}{\sqrt{e}} + \frac{1}{2}e^{-2x}\Big|_0^{1/2} - \frac{1}{e} + \frac{1}{\sqrt{e}} - \frac{1}{2e}\right) = 1 - \frac{2}{e} \qquad \approx 0.264$$

5.5. $X \sim \text{Unif}[0,1]$ $Y \sim \exp(\lambda)$

$Z = X + Y$

Compute $f_Z(z) = \int f_X(z-y) f_Y(y)\, dy$

If $0 \le z \le 1$, then $f_X(z-y) \neq 0$ for $0 \le y \le z$

If $z \geq 1$ then $f_X(z-y) \neq 0$ for $z-1 \le y \le z$

$$f_Z(z) = \begin{cases} \int_0^z f_X(z-y) f_Y(y)\,dy = \int_0^z \lambda e^{-\lambda y}\,dy = -e^{-\lambda y}\Big|_0^z = 1 - e^{-\lambda z} & 0 \le z \le 1 \\ \int_{z-1}^{z} \lambda e^{-\lambda y}\,dy = -e^{-\lambda y}\Big|_{z-1}^{z} = e^{-\lambda(z-1)} - e^{-\lambda z}, & z \geq 1 \end{cases}$$

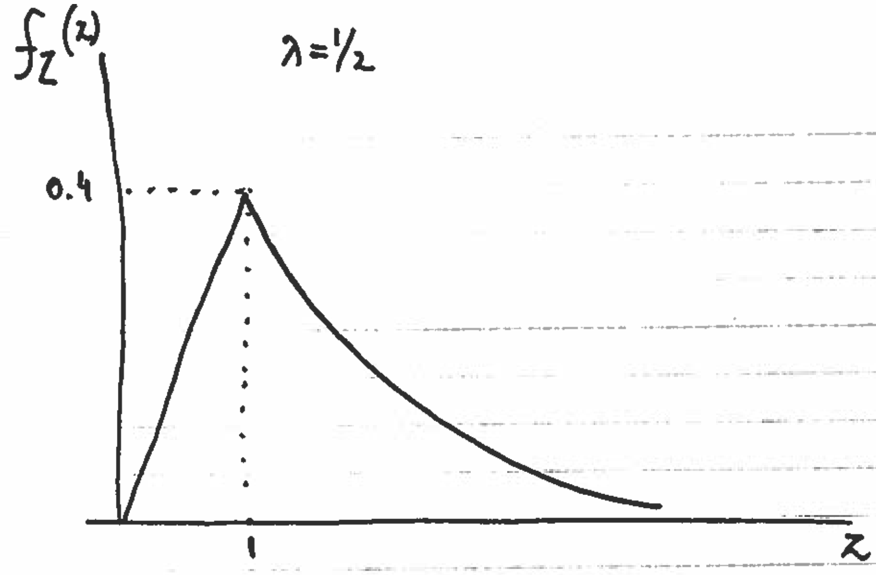
$$F_Z(z) = \int_0^z (1 - e^{-\lambda t})\,dt = z + \frac{1}{\lambda}(e^{-\lambda z} - 1) \qquad 0 \le z \le 1$$

For $z \geq 1$

$$F_Z(z) = \int_0^z f_Z(t)\,dt = F_Z(1) + \int_1^z (e^{-\lambda(t-1)} - e^{-\lambda t})\,dt$$

$$= 1 + \frac{1}{\lambda}(e^{-\lambda} - 1) + \left(-\frac{1}{\lambda}\right)\left(e^{-\lambda(t-1)} - e^{-\lambda t}\right)\Big|_1^z = 1 + \frac{1}{\lambda}(e^{-\lambda} - 1)$$

$$- \frac{1}{\lambda}(e^{-\lambda(z-1)} - e^{-\lambda z}) + \frac{1}{\lambda}(1 - e^{-\lambda}) = 1 - \frac{1}{\lambda}(e^{-\lambda(z-1)} - e^{-\lambda z}), \quad z \geq 1$$



(b) Compute the moment generating function

$$M_Z(s) = \mathbb{E}\left[e^{sZ}\right] = \int_0^\infty e^{sz} f_Z(z)\,dz = \int_0^1 e^{sz}\left(1-e^{-\lambda z}\right)dz$$

$$+ \int_1^\infty e^{sz}\left(e^{-\lambda(z-1)} - e^{-\lambda z}\right)dz \qquad \text{(assuming that } s<\lambda\text{)}$$

$$= \frac{1}{s}e^{sz}\Big|_0^1 - \frac{1}{s-\lambda}e^{(s-\lambda)z}\Big|_0^1 + \frac{e^\lambda}{s-\lambda}e^{(s-\lambda)z}\Big|_1^\infty - \frac{1}{s-\lambda}e^{(s-\lambda)z}\Big|_1^\infty$$

$$= \frac{e^s}{s} - \frac{1}{s} - \cancel{\frac{e^{s-\lambda}}{s-\lambda}} + \frac{1}{s-\lambda} - \frac{e^s}{s-\lambda} + \cancel{\frac{e^{s-\lambda}}{s-\lambda}} = -\frac{1-e^s}{s} + \frac{1-e^s}{s-\lambda} = -\frac{\lambda(1-e^s)}{s(s-\lambda)} \quad //$$

$(0<s<\lambda)$

(c) $\mathbb{E}[X^2] = \frac{1}{3}$, $\mathbb{E}[Y^2] = \frac{2}{\lambda^2}$

$$\mathbb{E}[Z^2] = \mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2+2XY+Y^2] \overset{\text{by independence}}{=} \mathbb{E}[X^2] + 2\,\mathbb{E}[X]\,\mathbb{E}[Y] + \mathbb{E}[Y^2]$$

$$= \frac{1}{3} + 2\cdot\frac{1}{2}\cdot\frac{1}{\lambda} + \frac{2}{\lambda^2} = \frac{1}{3} + \frac{\lambda+2}{\lambda^2} \quad //$$

5(a) $X \sim \text{Laplace}(2)$ $\quad f_X(x) = e^{-2|x|}, \quad -\infty < x < \infty$

$Y \sim \text{Unif}[0,2]$ $\quad f_Y(y) = \frac{1}{2} \quad 0 \le x \le 2$

(a) $P(Z > 2|Y|) \underset{\text{since } Y \geq 0 \text{ w.prob. } 1}{=} P(Z > 2Y) = P(X+Y > 2Y) = P(X > Y)$

$$= \int_{y=0}^{2} dy \int_{x=y}^{\infty} dx \cdot \frac{1}{2} e^{-2x} = \int_{y=0}^{2} dy \left(-\frac{1}{4} e^{-2x} \Big|_{y}^{\infty} \right) = \int_{0}^{2} \frac{1}{4} e^{-2y} dy = \frac{1}{8} - \frac{1}{8e^4} \approx 0.123$$

5(b) $f_Z(z) = (f_Y * f_X)[z] = \int f_Y(x) f_X(z-x)\, dx$

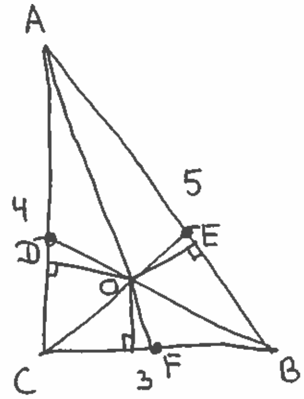
The integrand depends on the sign of $z - x$:

if $z - x < 0$ or $x > z$, then $f_X(z-x) = e^{-2(x-z)}$

Then

$$f_Z(z) = \begin{cases} \int_0^2 \frac{1}{2} e^{-2(x-z)} dx = -\frac{1}{4} e^{-2(x-z)} \Big|_0^2 = \frac{1}{4}\left(e^{2z} - e^{2(z-2)}\right), & z < 0 \\ \int_0^z \frac{1}{2} e^{2(x-z)} dx + \int_z^2 \frac{1}{2} e^{2(z-x)} dx = \frac{1}{4} e^{2(x-z)} \Big|_0^z + \frac{1}{4} e^{2(z-x)} \Big|_z^2 = -\frac{1}{4}\left(e^{-2z} - 2 + e^{-2(z-2)}\right) & 0 \le z \le 2 \\ \int_0^2 \frac{1}{2} e^{-2(z-x)} dx = +\frac{1}{4} e^{-2(z-x)} \Big|_0^2 = \frac{1}{4}\left(e^{-2(z-2)} - e^{-2z}\right), & z > 2 \end{cases}$$

Prob 7.



We need to find the area of the part of the triangle that is closer to the side AB.

The points that are closer to AB than to AC are located to the right of the bisector AF.

The same is true for the bisector BD.

The bisectors AB, BD, and CE intersect at the point O, which is the center of the inscribed circle in ABC.

There-fore, the perpendiculars from O to the sides AC, AB, and BC have the same length, call it h. We obtain

$$6 = \text{area}(ABC) = \text{area}(AOC) + \text{area}(COB) + \text{area}(AOB)$$
$$= \frac{h}{2}(3+4+5)$$

So $h=1$, and the area of the triangle AOB is $\frac{5}{2}$. This triange is exactly the set of points closer to the hypotenuse, so the probability in question is $\frac{5/2}{6} = \frac{5}{12}$ //

Prob. 8. We need to find the sign of $\rho(X,Y) = \frac{1}{\sigma_X \sigma_Y} \operatorname{cov}(X,Y)$. Compute

$$f_{XY}(x,y) = \begin{cases} 1/2 & \text{if } 0 \le x+y \le 2 \\ 0 & o/w \end{cases}$$

$$f_X(x) = \int_0^2 f_{X,Y}(x,y)\,dy = \begin{cases} \frac{2-x}{2} & 0 \le x \le 2 \\ 0 & o/w \end{cases}; \quad f_Y(y) = \begin{cases} 1-y/2 & 0 \le y \le 2 \\ 0 & o/w \end{cases}$$

$$EX = \int_0^2 x\left(1-\frac{x}{2}\right)dx = \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^2 = \frac{4}{2} - \frac{8}{6} = \frac{2}{3}; \text{ By symmetry } EY = \frac{2}{3}$$

$$E[XY] = \int_{x=0}^{2}\int_{y=0}^{2-x} \frac{xy}{2}\,dx\,dy = \int_{x=0}^{2} \frac{x}{2}\frac{(2-x)^2}{2}\,dx = \frac{1}{4}\left(\cancel{4}\frac{4x^2}{2} - 4\frac{x^3}{3} + \frac{x^4}{4}\right)_0^2 = \frac{1}{3}$$

$$\operatorname{cov}(X,Y) = E[XY] - EX\,EY = \frac{1}{3} - \frac{4}{9} = -\frac{1}{3}$$

Since $\sigma_X, \sigma_Y > 0$, we conclude that $\rho(X,Y) < 0$, so X and Y are negatively correlated //

HW6 ENEE 324

1. (a) $X \sim \mathcal{N}(0,1)$, $M_X(s) = e^{\frac{s^2}{2}}$

$$P(X \geq a) \leq \min_{s \geq 0} e^{-sa} M_X(s) = e^{-\max_{s\geq 0}(sa - \ln M_X(s))} = e^{-\max_{s\geq 0}(sa - \frac{s^2}{2})}$$

The maximum is attained for $s = a$, and we obtain

$$P(X \geq a) \leq e^{-a^2/2}$$

For $a = 2$ this gives $P(X \geq 2) \leq e^{-2} \cong 0.135$

The true value $P(X \geq 2) = 1 - \Phi(2) \cong 0.02275$

(b) Now $M_X(s) = e^{\lambda(e^s - 1)}$

$$P(X \geq c) \leq e^{-\max_{s\geq 0}(sc - \lambda(e^s - 1))} = e^{-(c \ln\frac{c}{\lambda} - (c-\lambda))}$$

The maximizing value of the argument $s = \ln\frac{c}{\lambda}$; this is ≥ 0 when $\lambda \leq c$.

The trivial bound $\underbrace{e^{-(c\ln\frac{c}{\lambda} - (c-\lambda))}}_{=1}$ is obtained when

$c \ln\frac{c}{\lambda} - (c-\lambda) \leq 0$ or $-\ln\frac{\lambda}{c} \geq 1 - \frac{\lambda}{c}$. This happens when $c = \lambda$.
(in the region $\lambda \leq c$)

6.2.
$$P(X=0) = \overset{0H}{\frac{1}{2^6}} + \overset{2H}{\binom{6}{2}} \frac{1}{2^6} + \overset{4H}{\binom{6}{4}} \frac{1}{2^6} + \overset{6H}{\binom{6}{6}} \frac{1}{2^6} = 2^{-6}(1+15+15+1) = \frac{1}{2}$$

Thus $p_X(k) = \begin{cases} 1/2 & k=0 \\ 1/2 & k=1 \end{cases}$

$$P(Y=0) = \overset{1H}{2^{-6}} + \overset{3H}{\binom{6}{3}} 2^{-6} + \overset{6H}{\binom{6}{6}} 2^{-6} = 2^{-6}(1+20+1) = \frac{22}{64}$$

$$\binom{6}{3} = \frac{6\cdot5\cdot4}{1\cdot2\cdot3}$$

$$P(Y=1) = 2^{-6}\Big(\underset{1H}{6} + \underset{4H}{\binom{6}{4}}\Big) = \frac{21}{64}$$

Thus $p_Y(k) = \begin{cases} 22/64 & k=0 \\ 21/64 & k=1 \\ 21/64 & k=2 \end{cases}$; $p_{X,Y}(i,j)$ $\begin{array}{c|cc} j \backslash i & 0 & 1 \\ \hline 0 & 2/64 & 20/64 \\ 1 & 15/64 & 6/64 \\ 2 & 15/64 & 6/64 \end{array}$

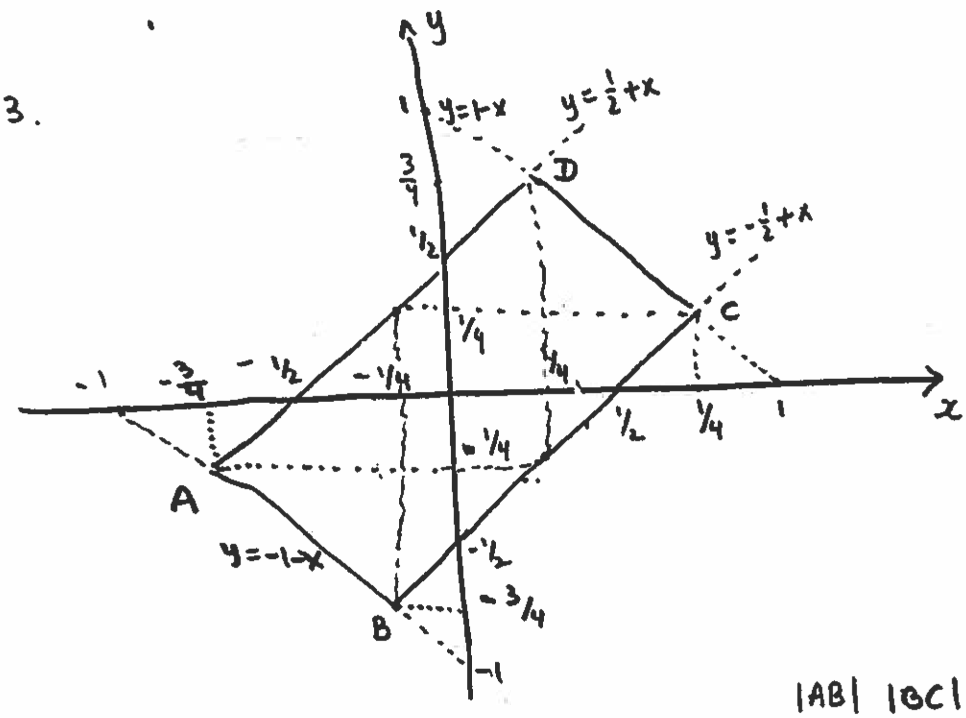
$$E[X] = \frac{1}{2} \qquad E[Y] = 0\cdot\frac{22}{64} + 1\cdot\frac{21}{64} + 2\cdot\frac{21}{64} = \frac{21}{64} + \frac{42}{64} = \frac{63}{64}$$

$$E[XY] = \sum_{i=0}^{1}\sum_{j=0}^{2} i\cdot j\; p_{X,Y}(i,j) = 1\cdot1\cdot\frac{6}{64} + 2\cdot1\cdot\frac{6}{64} = \frac{9}{32}$$

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{9}{32} - \frac{1}{2}\cdot\frac{63}{64} = \frac{9}{32} - \frac{15\tfrac{3}{4}}{32} < 0$$

X and Y negatively correlated, not independent

6.3.



Compute the area of the rectangle $= \overset{|AB|}{\frac{1}{2}} \cdot \overset{|BC|}{2} = 1$

Therefore, $f_{X,Y}(x,y)=1$ for $(x,y) \in \square$; 0 otherwise

Find
$$f_Y(y) = \begin{cases} \int_{-y-1}^{y+1/2} dx = y+1/2+y+1 = 2y+\frac{3}{2}, & -3/4 \le y \le -1/4 \\ 1 & -1/4 \le y \le 1/4 \\ \int_{y-1/2}^{1-y} dx = -2y+\frac{3}{2} & 1/4 \le y \le 3/4 \end{cases}$$

and 0 otherwise

$$f_X(x) = \begin{cases} \int_{-1-x}^{1/2+x} dy = 2x+3/2 & -3/4 \le x \le -1/4 \\ 1 & -1/4 \le x \le 1/4 \\ \int_{-1/2+x}^{1-x} dy = 1-x+\frac{1}{2}-x = -2x+\frac{3}{2} & 1/4 \le x \le 3/4 \\ 0 & x \le -3/4;\ x \ge 3/4 \end{cases}$$

(a) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$ for those y where $f_Y(y) \neq 0$ and undefined otherwise

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{2y+1.5} & -3/4 \le y \le -1/4 \\ 1 & -1/4 \le y \le 1/4 \\ \dfrac{1}{-2y+1.5} & 1/4 \le y \le 3/4 \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{1}{2x+1.5} & -3/4 \le \cancel{y}x \le -1/4 \\ 1 & -1/4 \le x \le 1/4 \\ \dfrac{1}{-2x+1.5} & 1/4 \le x \le 3/4 \end{cases}$$

(b) $$E[X|Y=y] = \int x\, f_{X|Y}(x|y)\,dx = \begin{cases} \displaystyle\int_{-1-y}^{1/2+y} \frac{x\,dx}{2y+1.5} = \frac{1}{2(2y+1.5)} x^2 \Big|_{-1-y}^{1/2+y} = -\frac{1}{4} & -\frac{3}{4} \le y \le -1/4 \\ y & -1/4 \le y \le 1/4 \\ 1/4 & 1/4 \le y \le 3/4 \end{cases}$$

$$E[X|Y] = \begin{cases} -\frac{1}{4} & -\frac{3}{4} \le Y \le -\frac{1}{4} \\ Y & -\frac{1}{4} \le Y \le \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \le Y \le \frac{3}{4} \end{cases}$$

$$E[X] = E[E[X|Y]] = \int f_Y(y)\, E[X|Y=y]\,dy = \int_{y=-3/4}^{-1/4} \left(-\frac{1}{4}\right)\left(2y+\frac{3}{2}\right) dy$$

$$+ \int_{-1/4}^{1/4} y\,dy + \int_{1/4}^{3/4} \left(\frac{1}{4}\right)\left(-2y+\frac{3}{2}\right) dy = 0$$

6.4. Let $X_1, X_2, X_3 \sim \exp(\lambda)$

$$Y = X_1 + X_2, \quad W = X_1 + X_2 + X_3 = Y + X_3$$

We have $f_{X_1}(x) = f_{X_2}(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

$$f_Y(y) = \int_0^y f_{X_1}(t) f_{X_2}(y-t)\,dt = \int_0^y \lambda^2 e^{-\lambda t - \lambda(y-t)}\,dt = \lambda^2 y e^{-\lambda y}, \; y \geq 0$$

$$f_W(w) = \int_0^w f_{X_3}(t) f_Y(w-t)\,dt = \int_0^w \lambda^3 \cancel{t}\, e^{-\lambda t} (w-t) e^{-\lambda(w-t)}\,dt$$

$$= \int_0^w \left[\lambda^3 w e^{-\lambda w} - \lambda^3 t e^{-\lambda w}\right] dt = \lambda^3 e^{-\lambda w} \int_0^w (w-t)\,dt = \lambda^3 e^{-\lambda w} \left(wt - \frac{t^2}{2}\right)\Big|_0^w$$

$$= \lambda^3 e^{-\lambda w} \left(w^2 - \frac{w^2}{2}\right) = \frac{\lambda^3}{2} e^{-\lambda w} w^2$$

Therefore $P(W \geq a) = \int_a^\infty \frac{\lambda^3}{2} e^{-\lambda w} w^2\,dw = \frac{\lambda^3}{2}\left[-\frac{1}{\lambda} e^{-\lambda w} w^2 \Big|_a^\infty + \frac{2}{\lambda} \int_a^\infty e^{-\lambda w} w\,dw\right]$

$$= \frac{\lambda^3}{2}\left[+\frac{1}{\lambda} e^{-\lambda a} \cdot a^2 + \frac{2}{\lambda}\left(-\frac{1}{\lambda} e^{-\lambda w} w \Big|_a^\infty + \frac{1}{\lambda}\int_a^\infty e^{-\lambda w}\,dw\right)\right]$$

$$= \frac{\lambda^3}{2}\left[+\frac{a^2}{\lambda} e^{-\lambda a} + \frac{2}{\lambda}\left(\frac{a}{\lambda} e^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda^2}\right)\right] = \left[\frac{(\lambda a)^2}{2} + \lambda a + 1\right] e^{-\lambda a}$$

We are given $\lambda = \frac{1}{2}$, $a = 20$, so $P(W \geq a) = (50 + 10 + 1) e^{-10} = 61 e^{-10} \approx 0.0027694$

At the same time, the Markov inequality gives $P(W \geq a) = \frac{6}{20} = 0.3$

5. (a) $EY_i = 2^{-i+1}$; $Var(Y_i) = \frac{9}{2^{2i}}$, $i = 1, 2, \ldots$

$$ET_n = \sum_{i=1}^{n} EY_i = \sum_{i=1}^{n} 2^{-i+1} = \frac{(\frac{1}{2})^n - 1}{-\frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right) = 2 - 2^{-n+1}$$

$$Var(T_n) = \sum_{i=1}^{n} Var Y_i = \sum_{i=1}^{n} \frac{9}{2^{+2i}} = \frac{9}{4} \sum_{i=0}^{n-1} \left(\frac{1}{4}\right)^i = \frac{9}{4} \, \frac{1-4^{-n}}{3/4} = 3(1-4^{-n})$$

$$E(A_n) = \frac{1}{n}(2 - 2^{-n+1}) \; ; \; Var(A_n) = \frac{3}{n^2}(1-4^{-n})$$

(b) We have $EY_i \to 0$, $Var(Y_i) \to 0$ as $i \to \infty$; by Chebyshev's inequality

$$P(|Y_i| \geq \varepsilon) \leq \frac{Var(Y_i)}{\varepsilon^2} = \frac{9}{2^{2i}\varepsilon^2} \xrightarrow[i\to\infty]{} 0 \text{. Thus } Y_i \xrightarrow[i\to\infty]{P} 0.$$

A similar argument shows that $A_n \xrightarrow[n\to\infty]{P} 0$.

The variables T_n do not converge anywhere.

ENEE324. Solutions to Probl. set 7

1.(a) $X_1 \perp\!\!\!\perp X_2$, $X_1 \sim \exp(\lambda)$ $\quad \mathbb{E}[X_1] = \frac{1}{\lambda} = 10$, so $\lambda = 1/10$

$X_2 \sim \exp(\lambda)$

$$f_{X_1}(x) = f_{X_2}(x) = 1/10\, e^{-1/10\, x}, \quad x \geq 0$$

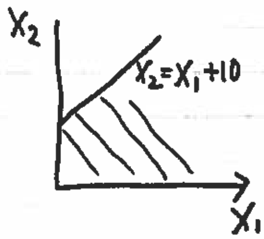
(b). $f_{X_1,X_2}(x_1,x_2) = \frac{1}{100} e^{-(x_1+x_2)/10}$, $\begin{matrix} x_1 \geq 0 \\ x_2 \geq 0 \end{matrix}$; 0 otherwise

by independence

(c) $$P(X_1 + X_2 \leq 20) = \int_{y=0}^{20} \int_{x=0}^{20-y} \frac{1}{100} e^{-\frac{x+y}{10}} dx\, dy = \int_{y=0}^{20} -\frac{1}{10} e^{-\frac{x+y}{10}} \Big|_0^{20-y} dy$$

$$= +\frac{1}{10} \int_0^{20} \left(e^{-\frac{y}{10}} - e^{-2}\right) dy = -e^{-y/10} \Big|_0^{20} - \frac{e^{-2} y}{10} \Big|_0^{20} = 1 - 3e^{-2} = 0.594$$

(d) $$P(X_2 - X_1 \leq 10) = \int_{y=0}^{\infty} \int_{x=0}^{y-10} \frac{1}{100} e^{-\frac{x+y}{10}} dx\, dy = \int_0^{\infty} -\frac{1}{10} e^{-\frac{x+y}{10}} \Big|_0^{y-10} dy$$



$$= -\frac{1}{10} \int_0^{\infty} \left(e^{-\frac{2y-10}{10}} - e^{-\frac{y}{10}}\right) dy = -\frac{1}{10} \left(-5e^{-\frac{2y-10}{10}} + 10e^{-\frac{y}{10}}\right)_0^{\infty}$$

$$= -\frac{1}{10}\left(\frac{5}{e} - 10\right) = 1 - \frac{5}{10e} = 0.816$$

(2) Compute mean time between arrivals:

$$\frac{1}{\lambda} = \frac{60\,ft}{30\ mph} = \frac{60}{30 \cdot 5280/3600s} = \frac{3600 \cdot 2}{5280} = \frac{16 \cdot 9 \cdot 25 \cdot 2}{3 \cdot 11 \cdot 16 \cdot 10} = \frac{15}{11}\ s$$

$$\lambda = \frac{11}{15}$$

(a) $T \sim \exp(\lambda) \quad f_T(x) = \frac{11}{15} e^{-\frac{11}{15}x}, \ x \geq 0$

(b) $\mathbb{P}(\text{1st arrival} \geq 2.5) = 1 - F_T(x)\big|_{x=2.5} = e^{-2.5\frac{11}{15}} \cong 0.160.$

(c) The two traffic flows are independent.
Let A be the event (cross the road w/out waiting)
A takes place if there are no arrivals within 2s
in both flows. Consider the two flows as a joint
Poisson process obtained by merging them. The arrival
rate $\mu = \lambda + \lambda = \frac{22}{15}$

$$\mathbb{P}(A) = \mathbb{P}(\text{1st arrival} \geq 2) = e^{-2 \cdot \frac{22}{15}} \cong 0.0532.$$

(3) (a) # arrivals is a binomial r.v.

$$P(4 \text{ arrivals}) = \binom{10}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^6 = \frac{210 \cdot 64}{3^{10}} \cong 0.228$$

(b) similarly $\binom{6}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 \approx 0.219$

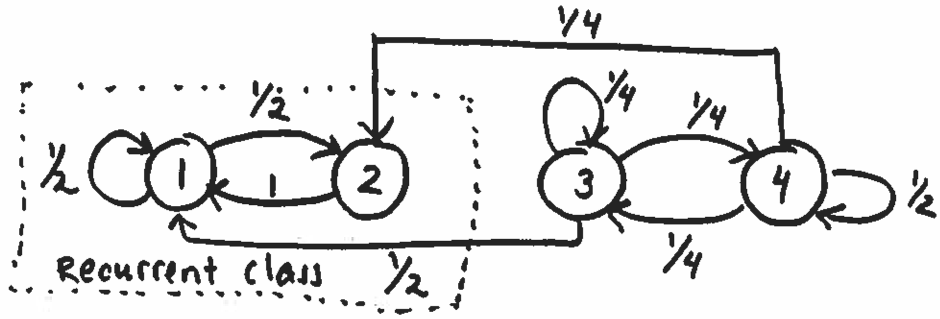
(c) similarly, by independence $\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{4}{81}$

(d) Pascal of order 3: $p(t) = \binom{t-1}{2}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{t-3}, \ t \geq 3 \ ; \ 0 \ \ t \leq 2.$

$\mathbb{E}[\text{3rd arr.}] = 9$

(e) $\mathbb{P}(\text{2 arrivals in 6 spots}) \cdot \frac{1}{3} = \binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 \cdot \frac{1}{3} = \frac{240}{2187} \approx 0.1097$

4. $\begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/2 \end{bmatrix}$



$\{1,2\}$ form an aperiodic recurrent class

$\{3,4\}$ are transient states

Steady-state distribution

$$\tfrac{1}{2}\pi_1 + \pi_2 + \tfrac{1}{2}\pi_3 = \pi_1$$

$$\tfrac{1}{2}\pi_1 + \tfrac{1}{2}\pi_4 = \pi_2$$

$$\tfrac{1}{4}\pi_3 + \tfrac{1}{4}\pi_4 = \pi_3$$

$$\tfrac{1}{4}\pi_3 + \tfrac{1}{2}\pi_2 = \pi_4$$

Solving this, we obtain

$$\pi = \left[\tfrac{2}{3}, \tfrac{1}{3}, 0, 0\right]$$

<u>Problem 5</u>: The chain has 13 states:

Colors	$\emptyset$	1	2	3	11	12	13	21	22	23	31	32	33
#	0	1	2	3	4	5	6	7	8	9	10	11	12

All transitions have prob. 0 or $1/3$. For instance from state 1 the chain can move to states 4, 5, 6 with prob. $1/3$

The transition matrix has the form

$$P = \begin{array}{cc} & \begin{array}{ccccccccccccc} 0&1&2&3&4&5&6&7&8&9&10&11&12 \end{array} \\ \begin{array}{c} 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12 \end{array} & \begin{bmatrix} &z&z&z&&&&&&&&& \\ &&&&z&z&z&&&&&& \\ &&&&&&&z&z&z&&& \\ &&&&&&&&&&z&z&z \\ &&&&z&z&z&&&&&& \\ &&&&&&&z&z&z&&& \\ &&&&&&&&&&z&z&z \\ &&&&z&z&z&&&&&& \\ &&&&&&&z&z&z&&& \\ &&&&&&&&&&z&z&z \\ &&&&z&z&z&&&&&& \\ &&&&&&&z&z&z&&& \\ &&&&&&&&&&z&z&z \end{bmatrix} \end{array}$$

where $z = \frac{1}{3}$ and the blank spots are filled with 0's.

States 0, 1, 2, 3 are <u>transient</u>, and the remaining 9 states are <u>recurrent</u>.

The stationary distribution is $\left(0,0,0,0,\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{1}{9}\right)$