ENEE 324 Solutions of HW2

1. Let $b(n,k,p) = \binom{n}{k} p^k (1-p)^{n-k}$ denote the binomial probability

$$P(\text{player 1 wins}) = 1 - b(6,0,\tfrac{5}{6}) = 1 - \left(\tfrac{5}{6}\right)^6 \approx 0.665$$

$$P(\text{player 2 wins}) = 1 - b(12,0,\tfrac{5}{6}) - b(12,1,\tfrac{5}{6}) \approx 1 - \left(\tfrac{5}{6}\right)^{12} - 12 \cdot \tfrac{1}{6} \cdot \left(\tfrac{5}{6}\right)^{11} \approx 0.619$$

Answer: player 1. //

2. Answer $\dfrac{\binom{90}{10}}{\binom{100}{10}} \approx 0.3305$ //

3. (a) Interesting question. If the marbles are indistinguishable, then there is just one way to partition them into pairs because every placement results in the same outcome.

If the marbles are numbered 1, 2, ..., 8, then there are

$$\frac{1}{4!}\binom{8}{2,2,2,2} = \frac{1}{4!}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{2520}{24} \text{ ways} = 105 \text{ ways}$$ //

Either answer is acceptable if supplied with correct reasoning.

(b) Every lineup has the same probability, so the needed quantity is

$$\frac{\#\text{ alternating lineups}}{\#\text{ all possible lineups}} = \frac{3 \cdot 3 \cdot 2 \cdot 2 \cdot 2}{6!} = \frac{72}{720} = 0.1$$ //

4. ~~Σ~~ Under the uniform distribution of quintuples the needed quantity is

$$\frac{\binom{15}{5}}{\binom{20}{5}} \approx 0.19$$

5. Let ~~...~~ us write the numbers as 000, 001, ..., 999. Let A_i be the subset of numbers that contain 1 in position $i = 1, 2, 3$.

The needed quantity $= |A_1| + |A_2| + |A_3| - |A_1 A_2| - |A_1 A_3| - |A_2 A_3| + |A_1 A_2 A_3|$
$= 3 \cdot 100 - 3 \cdot 10 + 1 = 271 = \#$ ~~...~~ numbers with at least one 1.

$$\text{Prob}(\text{Ramdom number contains} \geq \text{one } 1) = \frac{271}{1000} = 0.271$$

6 Let b = # blue balls in the first urn

r = # ~~blu~~ red balls in the first urn

By the total prob. theorem

$$P(\text{the chosen ball is blue}) = \frac{1}{2}\,\frac{b}{b+r} + \frac{1}{2}\,\frac{5-b}{(5-b)+(5-r)}$$

This number is maximal when $b=1, r=0$

and equals 0.722....

Answer: place a single blue ball and no red balls in one of the urns.

ENEE 324. Problem set 3

① Let $p = P(\text{a random point} \leq 1/4) = \frac{1}{4}$

Answer: $1 - 5p(1-p)^4 - (1-p)^5 = 1 - 5\frac{3^4}{4^5} - \frac{3^5}{4^5} \approx 0.367$

② $n = 3000$; $p = 0.0005$

The parameter of the Poisson distribution $\lambda = np = 1.5$

Prob (0 or 1 or 2 carry the mutation)

$$\approx e^{-1.5}\left(1 + \lambda + \frac{\lambda^2}{2}\right) = 3.365\, e^{-1.5} \approx 0.751$$

③ $P(1 \text{ or more} \geq 6) = 1 - P(\text{none are equal to } 6) = 1 - 0.9^n = 0.9$

$$0.9^n \leq 0.1 \Rightarrow n \geq 22$$

④ The probability that both players get the same number of H equals

$$\sum \binom{n}{i}^2 \left(\frac{1}{4}\right)^i \left(\frac{1}{4}\right)^{n-i} = 2^{-2n} \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n} 2^{-2n}$$

To compute $\sum \binom{n}{i}^2$ observe that

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2$$

⑤ Let F be the failure event. We know that

$$P(F|A) = 0.2 \qquad P(F|B) = 0.3 \qquad P(F|C) = 0.36$$

Use the Bayes formula to compute $P(C|F)$:

$$P(C|F) = \frac{P(F|C)P(C)}{P(F|A)P(A) + P(F|B)P(B) + P(F|C)P(C)}$$

$$= \frac{0.36 \cdot 0.2}{0.2 \cdot 0.8 + 0.3 \cdot 0.5 + 0.36 \cdot 0.2} = \frac{0.072}{0.16 + 0.15 + 0.072}$$

$$= \frac{0.072}{0.382} \approx 0.188$$

(6) $EX = \sum_{i=1}^{10} 0.1\pi i^2 = 0.1\pi \sum_{i=1}^{10} i^2 = 0.1\pi \frac{10\cdot 11\cdot 21}{6} = 38.5\pi$

$\cong 120.951$ //

(7) Suppose that the first time we have seen both H and T is n. This can happen only if the sequence of tosses up to (and including) toss n is

$$A_n = \begin{cases} TTT\ldots TH \\ HHH\ldots HT \end{cases}$$

Any other sequence of outcomes would have resulted in getting both H and T earlier than n.

The probability $P(A_n) = \frac{2}{2^n} = 2^{-n+1}$

$$E[N] = \sum_{n\geq 2} n P(A_n) = \cancel{2\left(\frac{1}{4}+\frac{1}{8}+\ldots\right)=\frac{1}{2}} -1 + \sum_{n=1}^{\infty} n \frac{2}{2^n}$$

$$= -1 + \frac{1}{(1-\frac{1}{2})^2} = 3 \,//$$

ENEE 324 HW4.

1.a. Let X be the RV equal to the number of cards before the 1^{st} ace.

$$P(X=i) = \underbrace{\frac{\binom{48}{i-1}}{\binom{52}{i-1}}}_{\text{no aces among the first } i-1 \text{ opened}} \underbrace{\frac{4}{52-(i-1)}}_{\text{an ace appears on } i\text{th trial}},$$

$$EX = \sum_{i=1}^{49} i\, P(X=i) \cong (\text{computer}) \boxed{10.6}$$

1b. Let X be defined as in part 1.a.

$X \sim \text{geom}(p)$, where $p = \frac{4}{52} = \frac{1}{13}$

$$P(X \geq 9) = \underbrace{\cancel{\tfrac{12}{13}}\,(1-p)^9}_{\text{at least 9 tries before the first success}} = \left(\frac{12}{13}\right)^9 \cong 0.49$$

2. $E[X(X-2)] = E[X^2-2X] = EX^2 - 2EX = EX^2 - 2 = 3$

$\Rightarrow EX^2 = 5$

$\text{Var}(X) = EX^2 - \underbrace{(EX)^2}_{=1} = 4$; $\text{Var}(-3X+5) = 9\,\text{Var}(X) = 36$

3. If the fellow is courteous enough to return the coats to the hanger, then his probability of success is $\frac{1}{n}$ every time, and the success is described by a geometric RV $(p = \frac{1}{n})$. The expectation $EX = n$, $\text{Var}(X) = \frac{1-p}{p^2} = n(n-1)$

If he's a true boor, and tosses the coats on the floor, then his prob of success in trial i is

$$P_1 = \frac{1}{n};\ P_2 = \frac{1}{n-1}\,\frac{n-1}{n} = \frac{1}{n};\ \dots;\ P_i = \underbrace{\frac{1}{n-(i-1)}}_{\text{finally, a success}} \times \underbrace{\frac{n-(i-1)}{n}}_{\text{no luck in the first } (i-1)} = \frac{1}{n}$$

Thus, his success time is uniformly distributed between $\{1, 2, \dots, n\}$, so

$$\boxed{E = \frac{n+1}{2};\ \ Var = \frac{n^2-1}{12}}$$

Use the provided "success" probability, i.e., $p = 0.03$ of taking more than 5 years.

(4.) Prob (2 or more out of 24 took more than 5 years)

$$= 1 - \binom{24}{0}(0.97)^{24} - \binom{24}{1} \times 0.03 \times 0.97^{23} \cong 0.162$$

Thus the claim that 3% take more than 5 years looks probable (the computed number is $< 50\%$). It is not fair to reject the claim.

(5.)

$$p_Y(0) = \sum_{x=0}^{1} p_{XY}(x,0) = \tfrac{1}{2}(1-p) + \tfrac{1}{2}p = \tfrac{1}{2}$$

$$p_Y(1) = 1 - p_Y(0) = \tfrac{1}{2}$$

$$p_{Y|X}(0|1) = \cancel{\frac{P_{XY}(1,0)}{P_X(0)} = \frac{\frac{1}{2}(1-p)}{1-P_X(1)} = \frac{\frac{1}{2}(1-p)}{\frac{1}{2}} = 1-p}$$

$$= \frac{P_{XY}(1,0)}{P_X(1)} = \frac{\frac{1}{2}\cancel{(1-p)}\,p}{P_{XY}(1,0) + P_{XY}(1,1)} = \frac{\frac{1}{2}p}{\frac{1}{2}p + \frac{1}{2}(1-p)} = p$$

$$p_{Y|X}(0|0) = \frac{P_{XY}(0,0)}{P_X(0)} = \frac{\frac{1}{2}(1-p)}{1 - p_X(1)} = \frac{\frac{1}{2}(1-p)}{\frac{1}{2}} = 1-p$$

(6) (We are assuming ~~bec~~ that X and Y are independent because we do not have any information to conclude otherwise)

(a) $$P(X+Y=10) = \sum_{i=1}^{9} P(X=i, Y=10-i) = \sum_{i=1}^{9} P(X=i)\,P(Y=10-i)$$

$$= \sum_{i=1}^{9} p(1-p)^{i-1} \times p(1-p)^{10-i-1} = \sum_{i=1}^{9} p^2(1-p)^8 = 9p^2(1-p)^8$$

(b) $$P_{X|X+Y}(i|10) = \frac{P(\{X=i\} \cap \{X+Y=10\})}{P(X+Y=10)} = \frac{P(\{X=i\})\,P(\{Y=10-i\})}{9p^2(1-p)^8}$$

$$= \frac{p^2(1-p)^8}{9p^2(1-p)^8} = \frac{1}{9} \quad \text{for all } i = 1, 2, \ldots, 9$$

$$P_{X|X+Y}(i|10) = 0 \quad \text{for all } i \leq 0 \text{ or } i \geq 10$$

ENEE 324 HW5, Solutions

1.

X	0	1	2	3	4

$$p_X(i) = \frac{\binom{30}{i}\binom{22}{4-i}}{\binom{52}{4}}, \quad i = 0,1,2,3,4$$

$$EX = \cancel{\frac{\binom{30}{1}\binom{22}{3}}{\binom{52}{4}}} \quad \frac{1}{\binom{52}{4}}\left[\binom{30}{1}\binom{22}{3} + 2\binom{30}{2}\binom{22}{2} + 3\binom{30}{3}\binom{22}{1} + 4\binom{30}{4}\right] =$$

$$\boxed{\approx 2.3079}$$

2. First, find a. $\sum_{n=0}^{\infty} \frac{(2t)^n}{n!} = e^{2t}$, so $a = e^{-2t}$

$$P(X<4) = p_X(0) + p_X(1) + p_X(2) + p_X(3) = e^{-2t}\left(1 + 2t + 2t^2 + \frac{4}{3}t^3\right)$$

$$P(X>1) = 1 - p_X(0) - p_X(1) = 1 - e^{-2t} - e^{-2t}\cdot 2t = 1 - e^{-2t}(1+2t)$$

3. $$\boxed{p_{N,X}(n,x) = p_N(n)\, p_{X|N}(x|n)}$$

The joint PMF is the product shown on this line, where the two multiples are computed below

where $\boxed{p_N(n) = \binom{t}{n} p^{t-n}(1-p)^n, \quad n = 0,1,\dots,t}$

$$p_{X|N}(x|n) = P(\text{all the failures occurred in locations } x+1,\dots,t \mid n \text{ failures})$$

We must consider the cases $n = 0,1,\dots,t$. If $n=0$ failures, then

$$\boxed{p_{X|N}(x|0) = \begin{cases} 1 & \text{if } x = t \\ 0 & \text{otherwise} \end{cases}}$$

If $n = 1,2,\dots,t$, then all the failures occur in locations $x+1,\dots,t$, and each such configuration arises with the same probability $\frac{\binom{t-x}{n}}{\binom{t}{n}}$

We obtain

$$\boxed{p_{X|N}(x|n) = \begin{cases} \frac{\binom{t-x}{n}}{\binom{t}{n}}, & x = 0,1,\dots,t-n \\ 0 & \text{otherwise.} \end{cases}}$$

4. $$p_{X|A_n}(k|A_n) = \frac{P(\{X=k\}\cap A_n)}{P(A_n)} = \frac{P(\{X=k\}\cap\{X+Y=n\})}{P(\{X+Y\}=n)} = \frac{P(\{X=k\})\,P(\{Y=n-k\})}{\sum_{i=1}^{n-1} p_X(i)\,p_Y(n-i)}$$

of course, k can be only one of the numbers $\{1,2,\dots,n-1\}$

We obtain

$$p_{X|A_n}(k|A_n) = \begin{cases} \dfrac{p(1-p)^{k-1}\;p(1-p)^{n-k-1}}{\sum_{i=1}^{n-1} p(1-p)^i\,p(1-p)^{n-1-i}} = \dfrac{1}{n-1} & \text{if } k=1,2,\dots,n-1 \\ 0 & \text{for all other } k \end{cases}$$

5. We have $\sin(X\pi)=0$ for $X=$ any integer

$\cos(X\pi)=(-1)^n$ if $X=n$

We obtain $\boxed{E[\sin(X\pi)]=0;\ E[\cos(X\pi)] = \sum_{n \text{ even}} p_X(n) - \sum_{n \text{ odd}} p_X(n)}$

6. Let X be the RV equal to the 1^{st} roll such that among the outcome of this and all the previous rolls each of the 6 numbers shows at least once.

Let $A_i^{(n)}=$ the event that number i did not show in the first n rolls.

$$P(A_i^{(n)}) = \left(\frac{5}{6}\right)^n$$

$$P(X>n) = P(A_1^{(n)}\cup A_2^{(n)}\cup\dots A_6^{(n)}) = \sum (-1)^j\,P(A_{i_1}\dots A_{i_j})$$

where the sum is on $j=1,\dots,5$ and all the combinations $i_1,\dots,i_j$, $1\le i_\ell\le 6$

We have $P(X>n) = \binom{6}{1}\left(\frac{5}{6}\right)^n - \binom{6}{2}\left(\frac{4}{6}\right)^n + \binom{6}{3}\left(\frac{3}{6}\right)^n - \binom{6}{4}\left(\frac{4}{6}\right)^n + \binom{6}{5}\left(\frac{5}{6}\right)^n$

$$p_X(n) = -P(X>n) + P(X>n-1) = \left(\frac{5}{6}\right)^{n-1} - 5\left(\frac{4}{6}\right)^{n-1} + 10\left(\frac{3}{6}\right)^{n-1} - 10\left(\frac{2}{6}\right)^{n-1} + 5\left(\frac{1}{6}\right)^{n-1}$$

for $n=6,7,\dots$

$p_X(n)=0$ for $n=1,2,3,4,5$

$$EX = \sum_{n=6}^{\infty} n\,p_X(n) \doteq \boxed{14.7} \leftarrow$$ compute this using the fact that for $|x|<1$ $$\sum_{n=6}^{\infty} n x^{n-1} = \frac{6x^5(1-x)+x^6}{(1-x)^2}$$

1. $f_X(x) = \frac{4}{\pi}, \quad 0 \le x \le \frac{\pi}{4}$

$$E[\cos(2X)] = \int_0^{\pi/4} \cos 2x \, f_X(x)\,dx = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \cos 2x\,dx = \frac{4}{\pi}\cdot\frac{1}{2}\int_0^{\pi/2}\cos t\,dt = \frac{2}{\pi}\sin t\Big|_0^{\pi/2} = \frac{2}{\pi}$$

$$E[\cos^2 X] = \frac{4}{\pi}\int_0^{\pi/4}\cos^2 x\,dx = \frac{4}{\pi}\int_0^{\pi/4}\left(\frac{1}{2}+\frac{\cos 2x}{2}\right)dx = \frac{1}{2}+\frac{1}{\pi}$$

2. $$F_X(x) = \begin{cases} 0 & -\infty < x < 4 \\ 1-\frac{16}{x^2} & 4 \le x < \infty \end{cases}$$

$$f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} 0 & -\infty < x < 4 \\ \frac{32}{x^3} & 4 \le x < \infty \end{cases}$$

$$EX = \int_4^\infty \frac{32\,dx}{x^2} = 32\cdot(-1)\frac{1}{x}\Big|_4^\infty = 8$$

$$Var(X) = 32\int_4^\infty \frac{dx}{x} = 32\ln x\Big|_4^\infty = \infty$$ (the integral diverges; Var X does not exist)

3. $$E[\ln X] = \int_1^2 \frac{2\ln x\,dx}{x^2} = -\frac{2\ln x}{x}\Big|_1^2 + \int_1^2\frac{2\,dx}{x^2} = 1-\ln 2$$

4. We find $EX = \int_0^1 (6x^2-6x^3)\,dx = \frac{1}{2}$; $EX^2 = \int_0^1 (6x^3-6x^4)\,dx = \frac{3}{10}$; $Var\,X = \frac{1}{20}$; $\sigma_X = \frac{1}{2\sqrt5}$

$$P\left(|X-EX| \le \frac{1}{2\sqrt5}\right) = \int_{\frac12-\frac{1}{2\sqrt5}}^{\frac12+\frac{1}{2\sqrt5}} (6x-6x^2)\,dx = (3x^2-2x^3)\Big|_{\frac12-\frac{1}{2\sqrt5}}^{\frac12+\frac{1}{2\sqrt5}} =$$

$$= 3\left(\left(\frac12+\frac{1}{2\sqrt5}\right)^2-\left(\frac12-\frac{1}{2\sqrt5}\right)^2\right) - 2\left(\left(\frac12+\frac{1}{2\sqrt5}\right)^3-\left(\frac12-\frac{1}{2\sqrt5}\right)^3\right)$$

$$= \frac{3\cdot2\cdot2}{4\sqrt5} - 2\cdot3\,\frac14\,\frac{1}{2\sqrt5} - 2\cdot\frac{1}{(2\sqrt5)^3} = \frac{6}{2\sqrt5} - \frac{3}{4\sqrt5} - \frac{2}{8\cdot5\sqrt5} = \frac{1}{\sqrt5}\left(3-\frac34-\frac{1}{20}\right) \boxed{= \frac{11}{5\sqrt5}}$$

5 The PDF of student's arrival at the bus stop is (counting from 8:45am)

$$f(x) = \begin{cases} 1/60 & 0 < x < 60 \\ 0 & o/w \end{cases}$$

To wait ≤ 10 min he needs to arrive between 8:50 and 9:00 ; or (9:20, 9:30)

to wait ≥ 15 min he needs to arrive either between 9:00 and 9:~~30~~15

or between 9:30 and 9:45

Thus the prob. for waiting ≤ 10 min is $\boxed{\frac{1}{3}}$; for waiting ≥ 15 min is $\boxed{\frac{1}{2}}$

6 We need $P(b^2 - 4 \geq 0)$ or $P(\{b > 2\} \cup \{b < -2\}) = \frac{1}{3}$ //

7 Let X be a random grade; $Y = \frac{1}{\sigma_X}(X - EX)$

$$P(X \geq 90) = P\left(Y \geq \frac{90-72}{\sqrt{7}}\right) = 1 - \Phi(6.8) \approx 5 \cdot 10^{-12}$$ (this must be a REALLY large class!)

~~$P(X \geq 80)$~~

$$P(80 \leq X \leq 90) = P(3.02 \leq Y < 6.8) = \Phi(6.8) - \Phi(3.02) \approx 0.00126$$

$$P(70 \leq X < 80) = P(-0.75 \leq Y < 3.02) = \Phi(3.02) - \Phi(-0.75) \approx 0.772$$

$$P(60 \leq X < 70) = P(-4.54 \leq Y < -0.75) = \Phi(-0.75) - \Phi(-4.54) \approx 0.227$$

$$P(X < 60) = P(Y < -4.54) = \Phi(-4.54) \approx 2.81 \cdot 10^{-6}$$

Problem set 7; solutions

(1) $F_X(x) = \begin{cases} 0 & x \le 0 \\ \int_0^x 4t^3\,dt = x^4 & 0 < x \le 1 \\ 1 & x > 1 \end{cases}$

$P(Y \le y) = P(1-3X^2 \le y) = P\left(\{X \ge \sqrt{\tfrac{1}{3}(1-y)}\} \cup \{X \le -\sqrt{\tfrac{1}{3}(1-y)}\}\right) = 1 - F_X\left(\sqrt{\tfrac{1}{3}(1-y)}\right)$

$$= \begin{cases} 0 & y \le -2 \\ 1 - \frac{1}{9}(1-y)^2 & -2 < y \le 1 \\ 1 & y \ge 1 \end{cases}$$

$f_Y(y) = F_Y'(y) = \frac{2}{9}(1-y)$ if $-2 \le y \le 1$ and $f_Y(y) = 0$ if $y < -2;\ y > 1$.

(2) A monitor will last < 15000 hrs if $X < 1.5$

(a) $P(X < \frac{1}{5}) = \int_1^{1.5} \frac{2}{x^2}\,dx = -2\frac{1}{x}\Big|_1^{1.5} = -\frac{4}{3} + 2 = \frac{2}{3}$

(b) $P(\{1 < X < 1.25\} | \{X < 1.5\}) = F_X(1.25)/F_X(1.5) = \frac{2}{5} \big/ \frac{2}{3} = \frac{3}{5}$

(3) $E[e^{aX}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ax - \frac{1}{2}x^2}\,dx = \frac{1}{\sqrt{2\pi}} e^{\frac{a^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{a^2}{2} + ax - \frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}} e^{\frac{a^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{2}}\,dx = e^{\frac{a^2}{2}}$ //

(the last integral $= 1$ because $e^{-(x-a)^2/2}/\sqrt{2\pi}$ is a PDF of $N(a,1)$)

To find the PDF of Y, take $t \ge 0$, then

$P(Y \le t) = P(-\sqrt{t} \le X \le \sqrt{t}) = 2\Phi(\sqrt{t}) - 1$

$f_Y(y) = \frac{d}{dy}(2\Phi(\sqrt{y}) - 1) = 2\frac{d}{dy}\int_{-\infty}^{\sqrt{y}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx = \frac{1}{\sqrt{2\pi y}} e^{-y/2}$ //

for $y \ge 0$ and 0 for $y < 0$.

(4) First, compute a: $1 = a\left(\frac{9}{2} - \frac{1}{2}\right) = 4a \Rightarrow a = \frac{1}{4}$

Next, $EX = \frac{1}{4}\int_1^3 x^2\,dx = \frac{1}{12}x^3\Big|_1^3 = \frac{13}{6}$; $\boxed{F_X(x) = \frac{x^2 - 1}{8},\ 1 \le x \le 3}$

$P(B) = P(1.7 \le X \le 2.7) = F_X(2.7) - F_X(1.7) = \frac{1}{8}(2.7^2 - 1.7^2) = \frac{1}{8}(7.29 - 2.89) = \frac{4.4}{8} = 0.55$

$E[X|B] = \int_{-\infty}^{\infty} x f_{X|B}(x)\,dx$; $f_{X|B}(x) = \frac{f_X(x)}{P(B)} = \frac{x}{2.2}$, $1.7 \le x \le 2.7$ and 0 o/w

$E[X|B] = \frac{1}{2.2}\int_{1.7}^{2.7} x^2\,dx = \frac{1}{6.6}(2.7^3 - 1.7^3) = \frac{19.683 - 4.913}{6.6} \simeq 2.24$ //

Problem set 7; solutions (Page 2)

⑤ We have (by independence) $f_{XY}(x,y) = f_X(x)\, f_Y(y) = 1 \cdot 2e^{-2y} = 2e^{-2y}$

$$P(Y \geq X) = \int_0^1 \int_x^\infty f_{XY}(x,y)\,dy\,dx = \int_0^1 \int_x^\infty 2e^{-2y}\,dy\,dx = \int_0^1 e^{-2x}\,dx = \tfrac{1}{2}\left(1-e^{-2}\right)$$

Let us find PDF $Z := X+Y$. If $Y=y$, then $Z = X+y$, i.e. $\cancel{f_{Z|Y}(z|y)}$

$$(\triangleright)\quad f_{Z|Y}(z|y) = f_X(z-y) = \begin{cases} 1 & \text{for } y \leq z \leq y+1 \\ 0 & \text{o/w} \end{cases}$$

Now let $Z=3$, then Y can be only between 2 and 3.

$$f_{Y|Z}(y|3) = \frac{f_{YZ}(y,3)}{f_Z(3)} = \frac{f_{Z|Y}(3|y)\, f_Y(y)}{f_Z(3)}$$

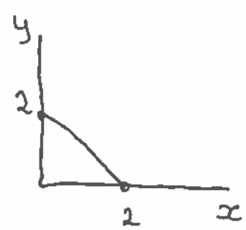
The PDF $f_{Z|Y}$ is computed above in $(\triangleright)$.

$$f_Z(3) = \int_2^3 f_{Z|Y}(3|y)\, f_Y(y) = \int_2^3 2e^{-2y}\,dy = e^{-4} - e^{-6}.$$

Finally
$$f_{Y|Z}(y|3) = \begin{cases} \dfrac{2e^{-2y}}{e^{-4}-e^{-6}} & 2 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

⑥

$X+Y \geq 2$



$$P(X+Y \geq 2) = \int_0^\infty \int_{\max(2-x,0)}^\infty \lambda_1\lambda_2 e^{-\lambda_1 x - \lambda_2 y}\,dy\,dx$$

$$= \int_0^2 \lambda_1 e^{-\lambda_1 x} \int_{2-x}^\infty \lambda_2 e^{-\lambda_2 y}\,dy\,dx + \int_2^\infty \lambda_1 e^{-\lambda_1 x} \int_0^\infty \lambda_2 e^{-\lambda_2 y}\,dy\,dx$$

$$= \lambda_1 \int_0^2 e^{-\lambda_1 x - \lambda_2(2-x)}\,dx + e^{-2\lambda_1} = \lambda_1 e^{-2\lambda_2} \int_0^2 e^{-(\lambda_1+\lambda_2)x}\,dx + e^{-2\lambda_1}$$

$$= e^{-2\lambda_1}\left[\frac{\lambda_1}{\lambda_1+\lambda_2}\left(1-e^{-2(\lambda_1+\lambda_2)}\right)+1\right]$$

$$P(\max(X,Y) \geq 2) = 1 - P(\{X \leq 2\} \cap \{Y \leq 2\}) = 1 - P(X \leq 2)\, P(Y \leq 2)$$

$$= 1 - \left(1-e^{-2\lambda_1}\right)\left(1-e^{-2\lambda_2}\right) = e^{-2\lambda_1} + e^{-2\lambda_2} - e^{-2(\lambda_1+\lambda_2)}$$

5(a) $X \sim \text{Laplace}(2)$ $\quad f_X(x) = e^{-2|x|}, \quad -\infty < x < \infty$

$Y \sim \text{Unif}[0,2]$ $\quad f_Y(y) = \frac{1}{2} \quad 0 \le x \le 2$

(a) $P(Z > 2|Y|) \underset{\text{since } Y \geq 0 \text{ w.prob. } 1}{=} P(Z > 2Y) = P(X+Y > 2Y) = P(X > Y)$

$$= \int_{y=0}^{2} dy \int_{x=y}^{\infty} dx \cdot \frac{1}{2} e^{-2x} = \int_{y=0}^{2} dy \left(-\frac{1}{4} e^{-2x} \Big|_y^{\infty} \right) = \int_0^2 \frac{1}{4} e^{-2y} dy = \frac{1}{8} - \frac{1}{8e^4} \approx 0.123$$

5(b) $f_Z(z) = (f_Y * f_X)[z] = \int f_Y(x) f_X(z-x)\, dx$

The integrand depends on the sign of $z - x$:

if $z - x < 0$ or $x > z$, then $f_X(z-x) = e^{-2(x-z)}$

Then

$$f_Z(z) = \begin{cases} \int_0^2 \frac{1}{2} e^{-2(x-z)} dx = -\frac{1}{4} e^{-2(x-z)} \Big|_0^2 = \frac{1}{4}\left(e^{2z} - e^{2(z-2)}\right), & z < 0 \\ \int_0^z \frac{1}{2} e^{2(x-z)} dx + \int_z^2 \frac{1}{2} e^{2(z-x)} dx = \frac{1}{4} e^{2(x-z)} \Big|_0^z + \frac{1}{4} e^{2(z-x)} \Big|_z^2 = -\frac{1}{4}\left(e^{-2z} - 2 + e^{-2(z-2)}\right) & 0 \le z \le 2 \\ \int_0^2 \frac{1}{2} e^{-2(z-x)} dx = +\frac{1}{4} e^{-2(z-x)} \Big|_0^2 = \frac{1}{4}\left(e^{-2(z-2)} - e^{-2z}\right), & z > 2 \end{cases}$$

Problem set 8, Solutions

① Let $X_i \sim \text{geom}\left(\frac{n-i+1}{n}\right)$, $i = 1, 2, \dots n$.

X_1 is the RV that is 1 if the first toss lands the marble into an empty box. Of course, $P(X_1 = 1) = 1$.

Now there are $n-1$ empty boxes left, and if X_2 is the RV ~~that is 1 if~~ equal to the number of the toss after toss 1 that ~~the second, or third, ... toss~~ lands the marble into an empty box, then

$$P(X_2 = j) = \left(\frac{1}{n}\right)^{j-1} \frac{n-1}{n}, \quad j \geq 1.$$

Generally, X_i is the RV equal to the number of the toss that takes out one more empty box, counting from the toss that ~~filled~~ resulted in $i-1$ nonempty boxes.

We are seeking the quantity

$$E(X_1 + X_2 + \dots + X_n) = 1 + \frac{n}{n-1} + \frac{n}{n-2} + \dots + \frac{n}{2} + n$$

② We have

$$\frac{2}{3} = \rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{1}{3}\,\text{cov}(X,Y)$$

$$\text{Cov}(X,Y) = 2$$

$$\text{Var}(3X - 5Y + 7) = 9\,\text{Var}(X) + 25\,\text{Var}(Y) - 15\,\text{Cov}(X,Y) = 204$$

Problem set 8, solutions (Page 2)

3. Let L be the random number of bits per letter.

$$p_L(n) = (1-p)^{n-1} p, \quad n = 1, 2, 3, \dots$$

$$P(L > n) = \sum_{\ell=n+1}^{\infty} p(1-p)^{\ell-1} = (1-p)^n, \quad n = 0, 1, 2, \dots$$

$$F_T(t) = P(T \le t) = P(L \le 1000t) = 1 - P(L > 1000t) = 1 - (1-p)^{1000t}$$

$$= 1 - e^{-t(-1000 \ln(1-p))}, \quad t \ge 0.$$

Answer: $T \sim \exp\left(1000 \ln \frac{1}{1-p}\right)$.

4. For $X = x$ we have $Y \sim \text{unif}[0, x]$, so

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 \le y < x \\ 0 & 1 \ge y > x \ge 0 \end{cases}$$

Since $f_X(x) = 1$, $0 < x < 1$, we obtain

$$f_{XY}(x,y) = f_{Y|X}(y|x) f_X(x) = \begin{cases} 1/x & 0 \le y < x, \ 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_y^1 \frac{dx}{x} = -\ln y \quad \text{for } 0 < y \le 1 \ ; \quad EY = -\int_0^1 y \ln y \, dy = -\frac{y^2}{2} \ln y \Big|_0^1 + \int_0^1 \frac{y\,dy}{2} = \frac{1}{4}$$

Using Law of Iterated expectations we obtain

$$EY = E_X\left[E[Y|X]\right] = E\frac{X}{2} = \frac{1}{4}$$

Area: $E[XY] = \int_0^1 \int_0^x xy \frac{1}{x} \, dy \, dx = \int_0^1 \frac{x^2}{2} dx = \frac{1}{6}$

(5) Since $M_X(t) = \dfrac{e^t - 1}{t} \quad (t \neq 0)$

$$M_Y(s) = E\left[e^{s(aX+b)}\right] = e^{bs} E e^{asX} = e^{bs} M_X(as)$$

$$= e^{bs} \frac{e^{as} - 1}{as} = \frac{e^{(a+b)s} - e^{bs}}{(a+b)s - bs}$$

We recognize $M_Y(s)$ as the transform of a uniform RV on $[b, a+b]$

(6) We have $M_X(s) = \left(\dfrac{e^s}{3} + \dfrac{2}{3}\right)^4$

so $X \sim \text{binom}\left(n=4, p=\frac{1}{3}\right)$

and $$P(X \leq 2) = \sum_{i=0}^{2} \binom{4}{i}\left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{4-i} = \left(\frac{2}{3}\right)^4 + 4 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^3 + 6 \cdot \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2$$

$$= \frac{16}{81} + \frac{32}{81} + \frac{24}{81} = \frac{72}{81} = \frac{8}{9}$$