324-16. Problem set 1

P1. (i) $A \cap B^c \cap C^c$

(ii) $A \cup B \cup C$

(iii) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$

(iv) $A \cap B \cap C$

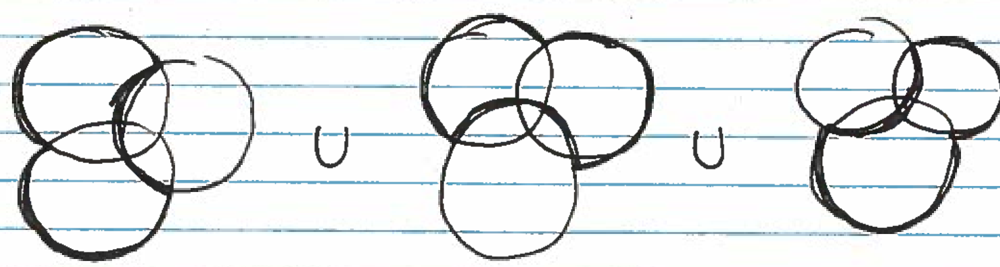
(v) $\Omega - (A \cup B \cup C) = A^c \cap B^c \cap C^c$

(vi) $((A \cup B) \cap C^c) \cup ((A \cup C) \cap B^c) \cup ((B \cup C) \cap A^c)$

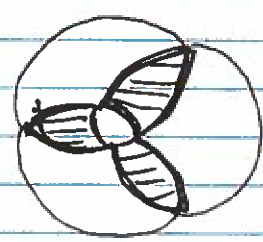
(vii) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$

Venn diagrams

(vi)



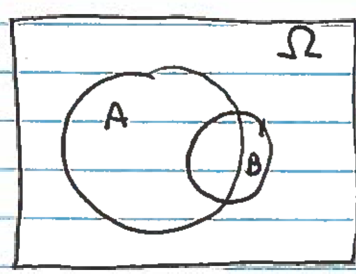
(vii)



The event in question is the dashed region

P2. Let $A = \{\text{flat tire}\}$, $B = \{\text{runs out of gas}\}$

$P(A) = 0.4$, $P(B) = 0.2$, $P(A \cap B) = 0.15$



$$P(A^c \cap B^c) = 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B \setminus A)$$

$$= 1 - 0.2 - (0.4 - 0.15) = 0.55$$

Problem 3.

Let A, B be as in Problem 2. We have 3 disjoint events:

$A \setminus B$, $B \setminus A$, $A \cap B$

By normalization

$$P(A\setminus B) + P(B\setminus A) + P(A\cap B) \leq 1 \qquad (1)$$

Next

$$P(A) + P(B) = P(A\setminus B) + P(B\setminus A) + 2P(A\cap B)$$

$$= P(A\setminus B) + P(B\setminus A) + P(A\cap B) + P(A\cap B) \leq 1 + P(A\cap B)$$

(the last step by (1))

This inequality implies that

$$P(A\cap B) \geq P(A) + P(B) - 1 = 0.5$$

Problem 4

Let A be the event that the snowbank melts away in a week.

$$P(A) = \frac{7}{14} = \frac{1}{2}$$

$$P(\underbrace{\text{melts in exactly 14 days}}_{\text{denote this event by } B}) = 1 - P(\text{13 days or fewer}) = \frac{1}{14}$$

$$P(B \mid A^c) = \frac{P(B\cap A^c)}{P(A^c)} = \frac{P(B)}{P(A^c)} = \frac{1/14}{1/2} = \frac{1}{7}$$

Problem 5

(a) $\underbrace{P(\text{1st ball red})}_{\text{call this event } R_1} = \frac{12}{18}$

Let R_2, R_3 be the events that 2^{nd} (3^{rd}) is red

$P(R_2 | R_1) = \frac{11}{17}$ — remaining # of red

total count of remaining balls

$P(R_3 | R_1 R_2) = \frac{10}{16}$

$P(R_1 R_2 R_3) = P(R_1) P(R_2 | R_1) P(R_3 | R_1 R_2) = \frac{12 \cdot 11 \cdot 10}{18 \cdot 17 \cdot 16} = \frac{11 \cdot 5}{3 \cdot 17 \cdot 8} = \frac{55}{408}$

(b) Let B_i, $i = 1, 2, 3$ be the event that the i^{th} ball is blue

The event in question is

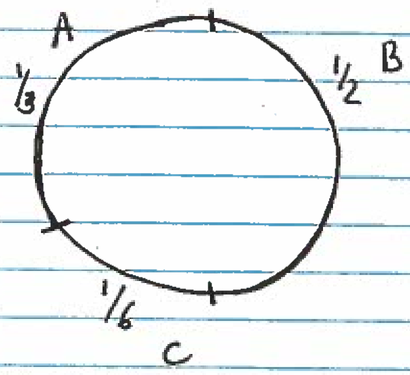
$$E = R_1 R_2 B_3 \cup R_1 B_2 R_3 \cup B_1 R_2 R_3$$

$$P(E) = P(R_1) P(R_2 | R_1) P(B_3 | R_1 R_2) + P(R_1) (P(B_2 | R_1) P(R_3 | R_1 B_2) + P(B_1) P(R_2 | B_1) P(R_3 | B_1 R_2)$$

$$= \frac{12}{18} \frac{11}{17} \frac{6}{16} + \frac{12}{18} \frac{6}{17} \frac{11}{16} + \frac{6}{18} \frac{12}{17} \frac{11}{16}$$

$$= \frac{12}{18} \frac{11}{17} \left(\frac{6}{16} + \frac{6}{16} + \frac{6}{16} \right) = \frac{6 \cdot 11}{8 \cdot 17} = \frac{66}{136} = \frac{33}{68}$$

Problem 6



$$P(A|C^c) = \frac{P(A \cap C^c)}{P(C^c)} = \frac{P(A)}{P(C^c)} =$$

$$= \frac{1/3}{5/6} = \frac{2}{5}$$

 Problem set 2 : Solutions

Problem 1. Let W be ~~the~~ a white ball and R the red one. The sample space

$$\Omega = \{ \underbrace{WW\ldots W}_{j}R \text{ for all } j=0,1,\ldots,n-1\}$$

Let $j=0$, then $P_1(R) = \frac{1}{n}$, where $P_m(R)$ is the probability of the red ball appearing in m^{th} drawing, $m=1,2,\ldots,n$.

$$P_2(R) = Pr(WR) = P_1(W)P_2(R|W) = \frac{n-1}{n}\,\frac{1}{n-1} = \frac{1}{n}$$

$$P_m(R) = \frac{n-1}{n}\,\frac{n-2}{n-1}\cdots\frac{n-(m-1)}{n-(m-2)}\,\frac{1}{n-(m-1)} = \frac{1}{n}, \quad m = \,.\,1,\ldots, n-1$$

Finally, $P_n(R) = 1$

Problem 2. Let n_i, $i=1,\ldots,5$ be the chosen numbers and let $A_i = \{n_i \geq 16\}$, $i=1,\ldots,5$. Use the multiplication rule:

$$P(A_1A_2\ldots A_5) = P(A_1)P(A_2|A_1)\ldots P(A_5|A_1A_2A_3A_4)$$

$$= \frac{15}{21}\,\frac{14}{20}\,\frac{13}{19}\,\frac{12}{18}\,\frac{11}{17} \cong 0.148$$

Another technique: There are $\binom{21}{5}$ possible choices of 5 numbers of which $\binom{15}{5}$ satisfy the condition of the problem. Since every choice has the same probability, the answer is given by

$$P(A_1A_2\ldots A_5) = \frac{\binom{15}{5}}{\binom{21}{5}},$$ which is the same as above

Problem 3 Let $P(H) = p$. The sample space Ω is the set of all infinite sequences $(a_1a_2\ldots)$, where each $a_i \in \{H,T\}$.

(a) the answer is

$$1 - P(\text{no H in the first n flips}) = 1 - 0.9^n \quad (\text{since } p=0.9)$$

(b) There are $\binom{n}{k}$ possibilities for k H's to appear among the first n outcomes, and each of them has probability $p^k(1-p)^{n-k}$

<u>answer</u>: $\binom{n}{k} 0.9^k (0.1)^{n-k}$

(c)-(d) The probability of n H's in succession is p^n, so for all $p \neq 1$. the answer is given by

$$\lim_{n\to\infty} p^n = 0.$$ Answer: 0 for both questions (c) and (d)

<u>Problem 4</u> The sample space $\Omega = \{ s_1 s_2 s_3 \}$ is formed of all the possible triples of students, order disregarded

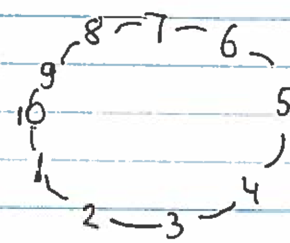
(a) There are $|\Omega| = \frac{60 \cdot 59 \cdot 58}{3!} = 34220$ possibilities

(b) $3 \cdot \left(\frac{20 \cdot 19 \cdot 18}{3!}\right) = 3 \cdot \binom{20}{3} = 3420$

(c) $3 \cdot \left(\frac{20 \cdot 19}{2}\right) \cdot 40 = 22800$

(d) $20^3 = 8000$

<u>Problem 5</u> Let



be the seats at the table.

~~The sample space Ω is~~

There are 10! possibilities to assign 10 persons to the seats. However, we are not distinguishing the order of the men or of the women because our question is whether the men are next to each other irrespective of how they are ordered. Therefore there are 10 different possibilities to seat them at the round table, starting with seat i, $i = 1, \dots, 10$. There are $\binom{10}{5}$ possibilities to assign seats to the group of 5 men (again order disregarded).

Answer $\frac{10}{\binom{10}{5}} = \frac{5}{126}$

Problem 6

(a) We are representing 8 as a sum of 4 nonnegative integers:

$$b_1 + b_2 + b_3 + b_4 = 8$$

where $0 \leq b_i \leq 8$, $i = 1,2,3,4$ is the number of marbles in Box i.

As explained in class, the number of possible choices for the tuple (b_1, b_2, b_3, b_4) is

$$\binom{8+4-1}{4-1} = \binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{2 \cdot 3} = 11 \cdot 5 \cdot 3 = 165$$

(b) In the first question, we are selecting boxes for the marbles $m_1, \dots, m_8$. The first marble m_1 can be ~~B_i~~ put into each of $B_1, \dots, B_4$, giving us 4 possibilities. The same for $m_2, m_3, \dots, m_8$, so the answer is $4^8 = 2^{16} = 65536$

In the second question, we are selecting two marbles for each of the boxes. There are 8 options ~~for~~ to select the first of the two marbles for B_1, 7 to select the second of the two for B_1, but since $\{m_1, m_2\}$ and $\{m_2, m_1\}$ account for the same allocation of marbles to B_1, ~~to~~ the number of choices for B_1 is $\frac{1}{2} 8 \cdot 7 = \binom{8}{2}$.

Answer: $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 2520$

324 Problem set 3. Solutions

Problem 1

(a) The first card can be chosen in any way; the second has a different denomination, i.e. there are 48 out of the remaining 51 possibilities; for the 3^{rd} there are 44 out of 50, and for the fourth 40 out of 49.

Answer: $\frac{48}{51} \cdot \frac{44}{50} \cdot \frac{40}{49} \approx \cancel{0.037}\ 0.676$

(b) Similarly, we obtain $\frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49} \cong 0.117$

(c) The value of 10^{th} card does not depend on earlier choices. So the answer is $\frac{4}{52} = \frac{1}{13}$

(d) Let $A = \{\text{card \#10 is Q}\}$; $B = \{\text{cards 1 through 9 aren't Q}\}$

$$P(AB) = P(A|B)\,P(B) = \frac{4}{43} \cdot \prod_{i=1}^{9} \frac{48-i}{52-i} \cong 0.042$$

Problem 2

(a) Only one of $\binom{36}{6}$ choices wins. Answer $\frac{1}{\binom{36}{6}} \approx 5.13 \cdot 10^{-7}$

(b) Let P_k be the probability of guessing k numbers, $1 \le k \le 6$

$$P_k = \frac{\binom{36-6}{6-k}\binom{6}{k}}{\binom{36}{6}}$$

So $P_3 = \binom{30}{3}\binom{6}{3} \Big/ \binom{36}{3} \approx 0.041$

(c) The answer is $P_3 + P_4 + P_5 + P_6$, where P_k is defined in Part (b)
≈ 0.045

Problem 3

We can first fix the outcome of roll 8, then the event that interests us is that it did not come up in rolls 1 through 7, So the answer is $\left(\frac{5}{6}\right)^7 \approx 0.279$

Problem 4: $x_5\, x_4\, x_3\, x_2\, x_1$

There are two possible cases: $x_5 = 0$ and $x_5 = 1$ (any value greater than 2 gives a number greater than 20000)

Case 1. $x_5 = 0$

$$x_4 + x_3 + x_2 + x_1 = 5$$

$$\# \text{ of cases} = \binom{5+4-1}{4-1} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3!} = 56$$

Case 2. $x_5 = 1$

$$x_4 + x_3 + x_2 + x_1 = 4$$

$$\# \text{ of cases} = \binom{7}{3} = 35$$

Prob. of the event $\{\text{sum of digits} = 5\} = \frac{56+35}{10^5} \approx 0.00091$

0.0045

20000

<u>Problem 5</u> The probability to get 3 or more black balls in 6 trials is

$$\sum_{i=3}^{6} \binom{6}{i} p^i (1-p)^{6-i}$$

and the probability to get 2 or more black balls in 4 trials is

$$\sum_{i=2}^{4} \binom{4}{i} p^i (1-p)^{4-i}$$

Comparing these two numbers, we find that for $0.6 < p < 1$ playing 6 rounds is to your gain (higher probability of the win)

~~<u>Problem 6.</u> The payoff is given by~~ $W = n - \frac{50-n}{2}$.

~~(a) If~~ $-1 \le W \le 1$ ~~then~~ $16 \le n \le 17.3$.

~~The probability of this even is given by~~

$$\binom{50}{16}\left(\frac{1}{4}\right)^{16}\left(\frac{3}{4}\right)^{34} + \binom{50}{17}\left(\frac{1}{4}\right)^{17}\left(\frac{3}{4}\right)^{33} \approx 0.108$$

~~(b) If~~ $-5 \le W \le 5$ ~~then~~ $13.3 \le n \le 20$, ~~and the probability of this event is~~

$$\sum_{i=14}^{20} \binom{50}{i}\left(\frac{1}{4}\right)^{i}\left(\frac{3}{4}\right)^{50-i} \approx 0.357$$

Please see next page

Problem 6. The payoff is given by $W = n - \frac{50-n}{2}$

(a) Let us compute $P(-1 < W < 1)$. This means that $16 < n < 17.3$, i.e., $n = 17$. The probability of this event is

$$\binom{50}{17}\left(\frac{1}{4}\right)^{17}\left(\frac{3}{4}\right)^{33} \cong 0.043$$

The probability of the event in question equals $1 - 0.043 \cong 0.957$

(b) Similarly if $-5 < W < 5$ then $13.3 < n < 20$, so the answer is given by

$$1 - \sum_{i=14}^{19} \binom{50}{i}\left(\frac{1}{4}\right)^{i}\left(\frac{3}{4}\right)^{50-i} \cong 0.349$$

ENEE 324, HW 4 Solutions

Problem 2 X takes values $-2,-1,0,1,2$. We observe that $X=-2$ only if $X_1=1$, $X_2=3$, so $p_X(-2)=\frac{1}{9}$. Analyzing all the possibilities, we obtain the pmf

$$p_X(k)=\begin{cases} 1/9 & \text{if } x=\pm 2 \\ 2/9 & \text{if } x=\pm 1 \\ 3/9 & \text{if } x=0 \\ 0 & \text{o/w} \end{cases}$$

$$EX=\sum_{k=-2}^{2} \cancel{k}\, p_X(k) = -2\cdot\frac{1}{9}-1\cdot\frac{2}{9}+0\cdot\frac{3}{9}+1\cdot\frac{2}{9}+2\cdot\frac{1}{9}=0$$

$$Var(X)=\sum_{k=-2}^{2}(k-0)^2 p_X(k)=\frac{4}{9}+\frac{2}{9}+0\cdot\frac{3}{9}+\frac{2}{9}+\frac{4}{9}=\frac{4}{3}$$

Problem 3: (a) The sample space is of the form

$$(a_1, a_2, \dots, a_8), \quad \text{where } \sum_{i=1}^{8} a_i = 4$$

and a_i is the number of cars in lane $i=1,2,\dots,8$

(b) answer: $\dfrac{8!}{a_1!\,a_2!\dots a_8!}\left(\dfrac{1}{8}\right)^4$

(c) $P_X(i)=\binom{4}{i}\left(\frac{1}{8}\right)^i\left(\frac{7}{8}\right)^{4-i}$, $i=1,2,3,4,0$ and $p_X(i)=0$ o/w

binomial because X is ~~a se~~ the number of successes (choices of lane 1) in a sequence of 4 independent Bernoulli experiments

Problem 4. Let B_i be the event that i does not come up during the first n rolls, $i=1,2,\dots,6$. Find

$$P(X>n) = P(B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6)$$

$$= \sum_{i=1}^{6} P(B_i) - \sum_{i,j} P(B_i B_j) + \sum_{i,j,k} P(B_i B_j B_k) - \sum_{i,j,k,l} P(B_i B_j B_k B_l)$$

$$+ \sum_{i,j,k,l,m} P(B_i B_j B_k B_l B_m)$$

$$= \binom{6}{1}\left(\frac{5}{6}\right)^n - \binom{6}{2}\left(\frac{4}{6}\right)^n + \binom{6}{3}\left(\frac{3}{6}\right)^n - \binom{6}{4}\left(\frac{2}{6}\right)^n + \binom{6}{5}\left(\frac{1}{6}\right)^n$$

$$= 6\left(\frac{5}{6}\right)^n - 15\left(\frac{4}{6}\right)^n + 20\left(\frac{3}{6}\right)^n - 15\left(\frac{2}{6}\right)^n + 6\left(\frac{1}{6}\right)^n$$

$$p_X(n) = P(X>n-1) - P(X>n)$$

$$= \left(\frac{5}{6}\right)^{n-1} - 5\left(\frac{4}{6}\right)^{n-1} + 10\left(\frac{3}{6}\right)^{n-1} - 10\left(\frac{2}{6}\right)^{n-1} + 5\left(\frac{1}{6}\right)^{n-1}, \quad n \geq 6$$

Problem 5 Find pmf $p_X(i)$, $i=1,2,3,4$

$$p_X(i) = \begin{cases} \dfrac{365}{365^4} & i=1 \\ \dfrac{\binom{4}{2}365\cdot 364 + \binom{4}{3}365\cdot 364}{365^4} & i=2 \\ \dfrac{\binom{4}{2}365\cdot 364\cdot 363}{365^4} & i=3 \\ \dfrac{365\cdot 364\cdot 363\cdot 362}{365^4} & i=4 \end{cases}$$

← (two have some common b/day and the other two some other common b/day; or three have a common b/day and the remaining one some other b/day)

$$EX = \sum_{i=1}^{4} i\, p_X(i) \simeq 3.98$$

Problem 5. We use the Poisson distribution with $\lambda = 0.025 \cdot 80 = 2$

Let A be the event in question, then

$$P(A) = 1 - P(0 \text{ golf fans}) - P(1 \text{ fan})$$
$$= 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!} = 1 - 3e^{-2} \approx 0.594$$

Problem 1:

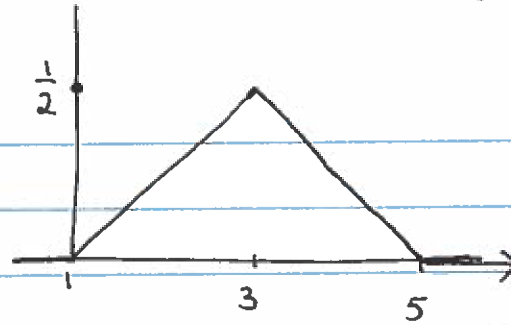
(a) $EX = \frac{1}{p} = 4$; $\quad P(X \leq 4) = \sum_{i=1}^{4} p_X(i) = \sum_{i=1}^{4} \left(\frac{3}{4}\right)^{i-1} \cdot \frac{1}{4}$

(b) $EX = 7p = 3.5 \quad P(X \leq 3.5) = \sum_{i=0}^{3} p_X(i) = \sum_{i=0}^{3} \binom{7}{i}\left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{7-i}$
$$= \left(\frac{1}{2}\right)^7 (1 + 7 + 21 + 35) = \frac{64}{128} = \frac{1}{2}$$

ENEE 324 HW5

1.

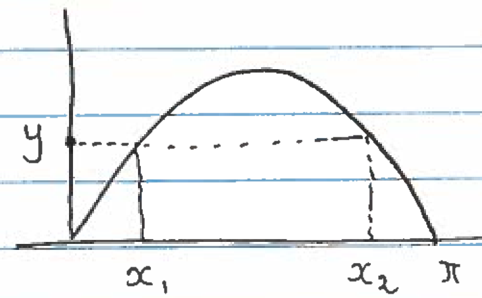
$$\varphi(x) = \begin{cases} \frac{1}{2} - \frac{1}{4}(3-x) & 1 \leq x \leq 3 \\ \frac{1}{2} - \frac{1}{4}(x-3) & 3 \leq x \leq 5 \end{cases}$$



$\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$, so it represents a valid pdf

$$F_X(x) = \begin{cases} 0 & -\infty < x \leq 1 \\ \int_1^x \left(\frac{1}{2} - \frac{1}{4}(3-t)\right)dt = \frac{x^2}{8} - \frac{x}{4} + \frac{1}{8}, & 1 < x \leq 3 \\ \cancel{-\frac{x^2}{8} + \frac{5x}{4} - \frac{9}{8}}, & 3 < x \leq 5 \quad -\frac{x^2}{8} + \frac{5x}{4} - \frac{17}{8} \\ 1 & x > 5 \end{cases}$$

2. ~~Range of Y: $0 < Y < 1$~~



$x_1 = \arcsin y$

$x_2 = \pi - x_1 = \pi - \arcsin y$

$$F_Y(y) = P(Y \leq y) = P(\{0 < X \leq x_1\} \cup \{x_2 \leq X < \pi\})$$

$$= \int_0^{x_1} \frac{2x}{\pi^2}\,dx + \int_{x_2}^{\pi} \frac{2x}{\pi^2}\,dx$$

$$= \frac{(\arcsin y)^2}{\pi^2} + 1 - \frac{(\pi - \arcsin y)^2}{\pi^2} = \frac{2\arcsin y}{\pi}$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} 0 & -\infty < y \leq 0 \\ \frac{2}{\pi\sqrt{1-y^2}} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

$$F_Z(z) = P(Z \leq z) = P\left(\sin\frac{X}{2} \leq z\right) = P\left(\frac{X}{2} \leq \arcsin z\right) = \int_0^{2\arcsin z} \frac{2x}{\pi^2}\,dx = \frac{4(\arcsin z)^2}{\pi^2}$$

$0 < x < \pi$

$0 < z < 1$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{8\arcsin z}{\pi^2\sqrt{1-z^2}}, \quad 0 < z < 1 \text{ and } 0 \text{ o/w}$$

(3.) We compute $EX = \int_0^1 (6x^2 - 6x^3)\,dx = \left(2x^3 - \frac{2}{3}x^4\right)\Big|_0^1 = \frac{1}{2}$

$$E(X^2) = \int_0^1 (6x^3 - 6x^4)\,dx = \left(\frac{6}{4}x^4 - \frac{6}{5}x^5\right)\Big|_0^1 = \frac{3}{10}$$

$$Var(X) = \frac{3}{10} - \frac{1}{4} = \frac{1}{20} \quad ; \quad \sigma_X = \frac{1}{2\sqrt{5}}$$

$$P\left(EX - 2\sigma_X < X < EX + 2\sigma_X\right) = P\left(\frac{1}{2} - \frac{2}{2\sqrt{5}} < X < \frac{1}{2} + \frac{2}{2\sqrt{5}}\right)$$

$$= \int_{\frac{1}{2} - \frac{1}{\sqrt{5}}}^{\frac{1}{2} + \frac{1}{\sqrt{5}}} (6x - 6x^2)\,dx = \left(3x^2 - 2x^3\right)\Big|_{\frac{1}{2} - \frac{1}{\sqrt{5}}}^{\frac{1}{2} + \frac{1}{\sqrt{5}}} = \frac{11}{5\sqrt{5}}$$

(4) (a) $P(X > t) = 1 - F_X(t) = e^{-\lambda t}$ //

(b) $P(a \le X \le b) = F_X(a) - F_X(b) = e^{-\lambda a} - e^{-\lambda b}$

$$P(a \le X \le t \mid X \le b) = \frac{P(a \le X \le t)}{P(X \le b)} = \frac{e^{-\lambda a} - e^{-\lambda t}}{\cancel{1 - e^{-\lambda b}}\ \ e^{-\lambda b} - e^{-\lambda a}} \text{ //}$$

$a \le X \le b$

(5) Compute

$$F_X(t) = P(X \le t) = \begin{cases} 0 & t < 0 \\ \frac{4t}{\pi} & 0 \le t < \frac{\pi}{4} \\ 1 & t \ge \frac{\pi}{4} \end{cases}, \quad f_X(t) = \frac{4}{\pi}, 0 < t < \frac{\pi}{4} \ ; \ 0 \text{ o/w}$$

$$E(\cos 2X) = \int_0^{\pi/4} \frac{4}{\pi} \cos 2x\,dx = \frac{2}{\pi} \sin 2x \Big|_0^{\pi/4} = \frac{2}{\pi} \text{ //}$$

$$E(\cos^2 X) = E\left(\frac{1}{2} + \frac{\cos 2X}{2}\right) = \frac{1}{2} + \frac{1}{\pi} \text{ //}$$

(6) We find $F_Y(y) = P(5X - 1 \le y) = P\left(X \le \frac{1}{5}(y+1)\right) = \begin{cases} 0 & y \le -1 \\ \frac{y+1}{5}, & -1 < y < 4 \\ 1 & y \ge 4 \end{cases}$

Conclude that $Y \sim \text{unif}\,[-1,4]$

ENEE324, HW6 (April 6, 2016)

Problem 1.(a) $F_Y(y) = P(Y \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\sqrt{y}}^{\sqrt{y}} e^{-\frac{(x-\mu)^2}{\sigma^2}} dx$

where $y \geq 0$

$F_Y(y) = 0$ if $y < 0$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{2\sqrt{2\pi y \sigma^2}} \left(e^{-\frac{(\sqrt{y}-\mu)^2}{2\sigma^2}} + e^{-\frac{(\sqrt{y}+\mu)^2}{2\sigma^2}} \right), \quad y \geq 0$$

0 o/w.

(b) $P(X>1) = 1 - \Phi\left(\frac{1-2}{\sqrt{3}}\right) \simeq 1 - \Phi(-0.58) = \Phi(0.58) \simeq 0.72$

$P(Y>1) = P(X<-1) + P(X>1)$

$P(X<-1) = \Phi(-\sqrt{3}) \approx 0.042$

$P(Y>1) \approx 0.72 + 0.042 = 0.762$.

Problem 2. Part (b) $F_{X/Y}(z) = P\left(\frac{X}{Y} \le z\right) = \iint_{\frac{x}{y} \le z} e^{-(x+y)} dx\, dy$

$$= \int_{y=0}^{\infty} \int_{x=0}^{zy} e^{-(x+y)} dx\, dy = \int_0^{\infty} (1 - e^{-zy}) e^{-y} dy = \left[-e^{-y} + \frac{e^{-y(z+1)}}{z+1} \right]_0^{\infty}$$

$$= 1 - \frac{1}{z+1}, \quad 0 < z < \infty$$

$$f_Z(z) = \frac{1}{(z+1)^2}, \quad 0 < z < \infty \text{ and } 0 \text{ o/w.}$$

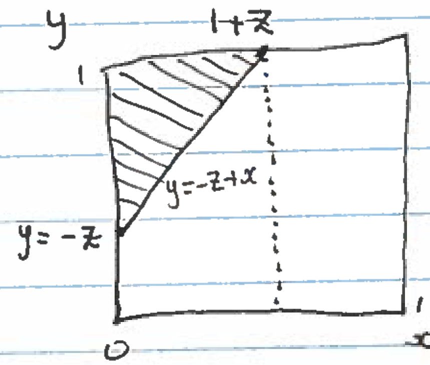
Part (a) (next page)

Problem 2 Part (a). The range of Z is $[-1,1]$

Find $F_Z(z) = P(X - Y \leq z)$, $-1 \leq z \leq 1$

(1) Let $z \leq 0$

Integrate over the region $y \geq -z + x$



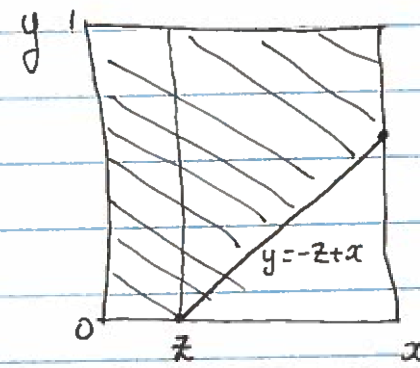
$$F_Z(z) = \int_{-z}^{1}\int_{0}^{y+z} 2x\,dx\,dy = \int_{-z}^{1} (y+z)^2\,dy$$

$$= \frac{(z+1)^3}{3}, \quad -1 < z \leq 0$$

(2) Let $z > 0$

Integrate over the region $y \geq -z + x$

or $x \leq y + z$



$$F_Z(z) = \int_0^z\int_0^1 2x\,dy\,dx + \int_z^1\int_{x-z}^1 2x\,dy\,dx$$

$$\cancel{= \int_0^z dy + \int^1 2x(1-z)\,dx = \frac{1}{3} + }$$

$$= z^2 + \int_z^1 2x\left(1-(x-z)\right)dx = -\frac{z^3}{3} + z + \frac{1}{3}, \quad 0 < z \leq 1$$

Answer:

$$F_Z(z) = \begin{cases} 0 & z \leq -1 \\ \dfrac{(z+1)^3}{3} & -1 < z \leq 0 \\ -\dfrac{z^3}{3} + z + \dfrac{1}{3} & 0 < z \leq 1 \\ 1 & z > 1 \end{cases}$$

Find PDF by differentiating

$$f_Z(z) = \begin{cases} (z+1)^2, & -1 < z \leq 0 \\ -z^2 + 1, & 0 < z \leq 1 \\ 0, & \text{o/w} \end{cases}$$

Problem 3 (a) use normalization:

$$\int_{-\infty}^{\infty} \lambda x e^{-x} dx = \lambda \left(-x e^{-x} \Big|_0^{\infty} + \int_0^{\infty} e^{-x} dx \right) = \lambda = 1 \quad \therefore \lambda = 1$$

(b) $$F(x) = \begin{cases} 0 & x < 0 \\ \int_{-\infty}^{x} \lambda t e^{-t} dt = \left[-(t+1) e^{-t} \right]_0^{x} = -(x+1)e^{-x}+1, & x \geq 0 \end{cases}$$

(c) $$P(3 \leq X \leq 6) = F(6) - F(3) = 4e^{-3} - 7e^{-6}$$

$$P(X \geq 8) = 1 - F(8) = 9e^{-8}$$

Problem 4. To find c, compute $\int_0^{\infty} \int_{-x}^{x} c e^{-x} dy\, dx = \int_0^{\infty} c(x-(-x)) e^{-x} dx$

$$= 2c \int_0^{\infty} x e^{-x} dx = 1 \quad \therefore c = 1/2$$

(b) $$f_X(x) = \int_{-x}^{x} \frac{1}{2} e^{-x} dy = \frac{1}{2x}, \quad -x < y < x \ ; \ 0 \ \text{o/w}$$

x exp(-x)

$$f_Y(y) = \int_{|y|}^{\infty} \frac{1}{2} e^{-x} dx = \frac{1}{2} e^{-|y|}, \quad -\infty < y < \infty$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = e^{-x+|y|}, \quad |y| < x < \infty$$

$$f_{Y|X}(y|x) = \frac{1}{2x}, \quad |y| < x$$

(c) Suppose x is fixed, then $f_{Y|X}(y|x)$ is uniform on $[-x, x]$, so

$$E[Y|X=x] = 0 \ ; \quad Var(Y|X=x) = \frac{(x-(-x))^2}{12} = \frac{x^2}{3}$$

Problem 5

$X \quad \lambda_1 \quad$ exponential

$X \perp\!\!\!\perp Y$

$Y \quad \lambda_2 \quad$ exponential

$Z = \min(X, Y)$

If $\min(X,Y) > z$, then both $X > z$ and $Y > z$

$$P(Z \leq z) = 1 - P(Z > z) = 1 - P(X > z, Y > z) = 1 - P(X > z)P(Y > z)$$

$$= 1 - (1 - F_X(z))(1 - F_Y(z)) = 1 - e^{-(\lambda_1 + \lambda_2)z}, \quad z \geq 0$$

$$F_Z(z) = \begin{cases} 0 & z < 0 \\ 1 - e^{-(\lambda_1 + \lambda_2)z}, & z \geq 0 \end{cases}$$

ENEE 324, HW7, Solutions. April 13, 2016

Problem 1. (a) Since $f_{Y|X}(y|x) = \frac{1}{x}$, $0<y<x$ and 0 o/w, we obtain

$$f_{XY}(x,y) = f_X(x) f_{Y|X}(y|x) = \begin{cases} e^{-x}, & 0<y<x \\ 0 & \text{o/w} \end{cases}$$

(b) $$EX = \int_0^\infty x^2 e^{-x} dx = -x^2 e^{-x}\Big|_0^\infty + \int_0^\infty 2x e^{-x} dx = -2xe^{-x}\Big|_0^\infty + 2\int_0^\infty e^{-x} dx = 2$$

$$EX^2 = \int_0^\infty x^3 e^{-x} dx = 3\int_0^\infty x^2 e^{-x} dx = 6$$

$$\text{Var } X = 6-4 = 2$$

(c) Find $$f_Y(y) = \int_{-\infty}^\infty f_{XY}(x,y)dx = \int_y^\infty e^{-x} dx = \begin{cases} e^{-y}, & y>0 \\ 0 & \text{o/w} \end{cases}$$

Thus $Y \sim \exp(1)$, so $E\cancel{X}Y = \text{Var}(Y) = 1$

(d) $$M_s(x) = \int_0^\infty e^{-x+sx} x\,dx = \int_0^\infty x e^{-(1-s)x} dx = \frac{1}{(1-s)^2}, \quad s<1$$

Sol. 1

Write a power series

$$M_s(x) = (1-s)^{-2} = \sum_{k=0}^\infty \binom{-2}{k}(-s)^k = \sum \frac{(k+1)!}{k!} s^k, \quad s<1$$

Thus $\boxed{EX^n = (n+1)!}$

Sol. 2 If you do not want to use the binomial formula for the power series, then compute

$$\frac{d}{ds} M_X(s) = 2(1-s)^{-3}; \quad \frac{d^2}{ds^2} M = 2\cdot 3(1-s)^{-4}; \ldots; \frac{d^n}{ds^n} M_X(s) = (n+1)!(1-s)^{-n}$$

Substituting $s=0$, we obtain $EX^n = \frac{d^n}{ds^n} M_X(0) = (n+1)!$

Problem 1 (e)

$$E(XY) = \int_0^\infty\int_0^x xy\, f_{XY}(x,y)\,dx\,dy = \int_0^\infty \int_0^x xy e^{-x}\,dy\,dx = \frac{1}{2}\int_0^\infty x^3 e^{-x}\,dx$$

$$= \frac{3}{2}\int_0^\infty x^2 e^{-x}\,dx = 3$$

$$Cov(X,Y) = E(XY) - (EX)(EY) = 3 - 2 = 1$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \ \sigma_Y} = \frac{1}{\sqrt{2}\cdot 1} = \frac{1}{\sqrt{2}}$$

Problem 2 (a) We have $M_X(s) = M_Y(s) = (pe^s + 1 - p)^n$, so

$M_Z(s) = (pe^s + 1 - p)^{2n}$. Therefore, $Z \sim Binom(2n, p)$

(b) We have $M_X(s) = e^{\lambda_1(e^s - 1)}$, $M_Y(s) = e^{\lambda_2(e^s-1)}$, so

$M_Z(s) = e^{(\lambda_1+\lambda_2)(e^s-1)}$, i.e., $Z \sim Poisson(\lambda_1 + \lambda_2)$

$$P_{X|Z}(k|n) = \frac{P(X=k)\,P(Y=n-k)}{P(Z=k)} = \frac{e^{-\lambda_1}\lambda_1^k}{k!}\,\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\,\frac{n!}{e^{-\lambda_1+\lambda_2}(\lambda_1+\lambda_2)^n}$$

$$= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k} = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(1 - \frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-k}$$

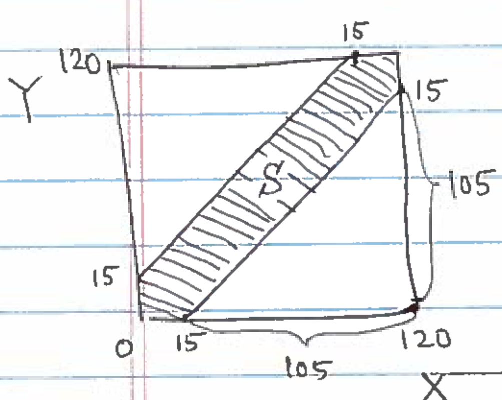
Thus, conditional on Z, $X \sim Binom\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)$

Problem 3. Let X, Y be the random arrival times, then

$$X \sim \text{unif}\,[0,120]\;;\; Y \sim \text{unif}\,[0,120]\;;\; f_{XY}(x,y) = \frac{1}{(120)^2}$$

We are interested in the event $A = \{|X-Y| \le 15\}$, equivalently

$$Y-15 \le X \le Y+15$$



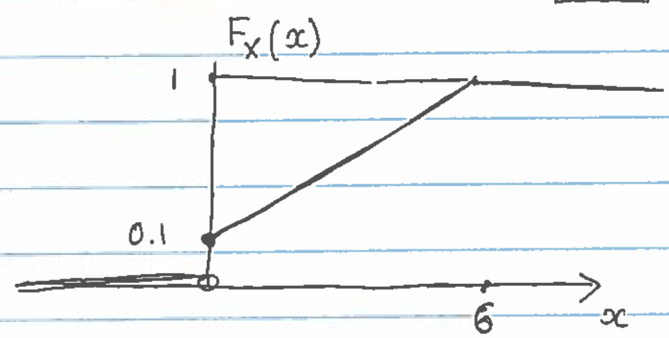
$P(A) = \frac{1}{(120)^2} \cdot$ area of the region between the diagonal segments

This area equals $(120)^2 - (105)^2$, so

$$P(A) = 1 - \left(\frac{105}{120}\right)^2 = \boxed{\frac{15}{64}}$$

Problem 4

(a)



$$f_X(x) = F_X'(x) = \begin{cases} 0 & x<0 \\ 0.1\delta(x) + 0.15, & 0 \le x < 6 \\ 0 & x \ge 6 \end{cases}$$

(b)

$$F_Y(x) = P(X^2 \le x) = P(X \le \sqrt{x}) = \begin{cases} 0 & x<0 \\ 0.1 + 0.15\sqrt{x} & 0 \le x < 36 \\ 1 & x \ge 36 \end{cases}$$

$$f_Y(x) = \begin{cases} 0 & x<0 \text{ or } x \ge 36 \\ 0.1\delta(x) + 0.075\frac{1}{\sqrt{x}}, & 0 \le x < 36 \end{cases}$$

$$\cancel{EY = \int_0^6 x^2(0.1\delta(x) + 0.15)\,dx = 0.05x^3\Big|_0^6 = 108}$$

$$EY = \int_0^{36} x\left(0.1\delta(x) + 0.075\frac{1}{\sqrt{x}}\right)dx = 0.075 \cdot \frac{2}{3}x^{3/2}\Big|_0^{36} = 0.05 \cdot 219 = 10.9$$

Problem 5

(a) Suppose we are given x, $1 \le x \le 2$

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{2-x}, & x \le y < 2 \\ 0 & 1 \le y < x \le 2 \end{cases}$$

$$f_{XY}(x,y) = f_{Y|X}(y|x)\underbrace{f_X(x)}_{=1,\ 1\le x\le 2} = \begin{cases} \frac{1}{2-x}, & x<y\le 2 \text{ and } 1\le x<2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_1^y \frac{dx}{2-x} = -\ln(2-y)\Big|_1^y = -\ln(2-y), \quad 1 \le y < 2;\ 0 \text{ o/w}$$

$$EY = -\int_1^2 y\ln(2-y)\,dy = \frac{7}{4} \quad \left(\begin{array}{l}\text{integrating by parts, we obtain} \\ -\int y\ln(2-y)dy = y + \frac{1}{4}y^2 + 2\ln(2-y) - \frac{1}{2}y^2\ln(2-y)\end{array}\right)$$

(b) $$EY = E_X[E(Y|X)] = E_X\left(\frac{X+2}{2}\right) = E\frac{X}{2}+1 = \frac{3/2}{2}+1 = \frac{7}{4}$$

(c) $$E[XY] = \int_1^2\int_x^2 \frac{xy}{2-x}\,dy\,dx = \int_1^2 \frac{x}{2-x}\frac{y^2}{2}\Big|_x^2 dx = \frac{1}{2}\int_1^2 \frac{x(4-x^2)}{2-x}dx$$

$$= \int_1^2 (2x+x^2)dx = \frac{1}{2}\left(2+\frac{8}{3}\right) = \frac{7}{3}$$

ENEE HW8 Solutions

Problem 1. Let $\mu_X = EX$; $\mu_Y = EY$, and let σ^2 be the common variance of X and Y. Let ρ_{XY} be the correlation coeff. of XY. Denote

$$U = X+Y \qquad V = X-Y \qquad \text{then} \qquad X = \tfrac{1}{2}(U+V) \quad Y = \tfrac{1}{2}(U-V). \qquad \text{Denote} \quad g_1(u,v) = \tfrac{1}{2}(u+v) \quad g_2(u,v) = \tfrac{1}{2}(u-v)$$

We show that

① U and V are uncorrelated

② U and V are jointly Gaussian

Together ① and ② imply that $U \perp\!\!\!\perp V$.

① We have

$$Cov(X+Y, X-Y) = E[(X+Y-(\mu_X+\mu_Y))(X-Y-(\mu_X-\mu_Y))]$$

$$= EX^2 - \mu_X^2 - EY^2 + \mu_Y^2 = Var(X) - Var(Y) = 0, \text{ so } \rho_{UV} = 0$$

② Let $f_{XY}(x,y) = \dfrac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\dfrac{1}{2(1-\rho^2)}\left[\dfrac{x^2}{\sigma^2} - \dfrac{2\rho xy}{\sigma^2} + \dfrac{y^2}{\sigma^2}\right]\right]$ be the joint PDF of X and Y, where for simplicity we take $\mu_X = \mu_Y = 0$.
(Textbook p.254-256)

Then

$$f_{UV}(u,v) = f_{XY}(g_1(u,v), g_2(u,v))\,|J|,$$

where

$$J = \begin{bmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -1, \text{ so } |J| = 1.$$

Thus

$$\ln\left(2\pi\sigma^2\sqrt{1-\rho^2}\, f_{UV}(u,v)\right) = -\frac{1}{2(1-\rho^2)}\left[\frac{(u+v)^2}{4\sigma^2} - \frac{2\rho(u+v)(u-v)}{\sigma^2} + \frac{(u-v)^2}{4\sigma^2}\right]$$

$$= -\frac{1}{2}\left[\frac{u^2}{2(2\sigma^2(1+\rho))} + \frac{v^2}{2(2\sigma^2(1-\rho))}\right]$$

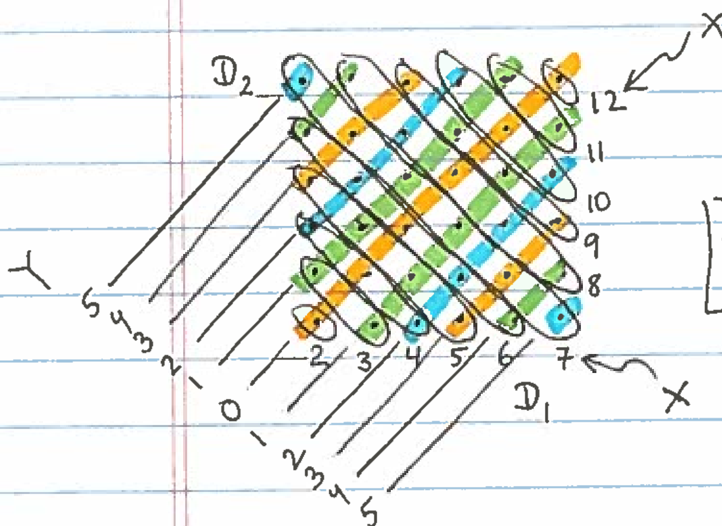
so $U \sim \mathcal{N}(0, 2\sigma^2(1+\rho))$ and $V \sim \mathcal{N}(0, 2\sigma^2(1-\rho))$ are indeed Gaussian

<u>Problem 2</u>. The $\boxed{\text{PMFs}}$ of X and Y are found to be

	2	3	4	5	6	7	8	9	10	11	12
X : $\frac{1}{36}\times$	1	2	3	4	5	6	~~7~~ 5	~~8~~ 4	~~9~~ 3	~~10~~ 2	1

	0	1	2	3	4	5
Y : $\frac{1}{36}\times$	6	10	8	6	4	2

The set of possible outcomes is shown schematically in the figure



Joint PMF of X, Y

Y \ X	2	3	4	5	6	7	8	9	10	11	12
0	1		1		1		1		1		1
1		2		2		2		2		2	
2			2		2		2		2		
3				2		2		2			
4					2		2				
5						2					

$\times \frac{1}{36}$

(blank spots are 0's)

$$\underline{EX} = \frac{1}{36}(2+6+12+20+30+42+40+36+30+22+12) = \frac{252}{36} = \underline{7}$$

$$\underline{EY} = \frac{1}{36}(10+16+18+16+10) = \frac{70}{36} = \underline{\frac{35}{18}}$$

$$\underline{E(XY)} = \frac{2}{36}(3+5+7+9+11+8+12+16+20+15+21+27+24+32+35) = \underline{\frac{245}{18}}$$

$$\underline{Cov(X,Y)} = E(XY) - EX \cdot EY = \frac{1}{18}(245-245) = \underline{0}$$

X and Y are <u>not independent</u> because

$$p_{XY}(3,1) = \frac{1}{18} \quad \text{but} \quad p_X(3)\,p_Y(1) = \frac{2}{36}\cdot\frac{10}{36} = \frac{5}{9}$$

Problem 3. As discussed in class (also textbook Sect. 5.5), for SLLN to hold, we need that

$$E X_i^4 < \infty$$

We have $E X_i^4 = \int_0^1 4x^5(1-x)\,dx = 4\left(\frac{1}{6} - \frac{1}{7}\right) = \frac{2}{21}$

so SLLN applies. We find $E X_i = \frac{1}{3}$, so

$$P\left(\lim_{n\to\infty} S_n = \frac{1}{3}\right) = 1.$$

Problem 4. We have n random samples $X_1, \dots, X_n$. Let $m \doteq EX$; $\sigma^2 \doteq Var(X)$.

Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Our task is to find n such that

$$P(|\overline{X}_n - m| < 2\sigma) \geq 0.98$$

or $$P(|\overline{X}_n - m| \geq 2\sigma) < 0.2.$$

Use Chebyshev's inequality to compute

$$P(|\overline{X}_n - m| \geq 2\sigma) \leq \frac{\sigma^2}{(2\sigma)^2 n} = \frac{1}{4n}$$

If $n \geq 12.5$ then $\frac{1}{4n} \leq 0.02$, so a sample of $n = 13$ suffices.

Problem 5. Compute $E X = \int_1^3 \frac{x}{9}\left(x + \frac{5}{2}\right) dx = \left(\frac{x^3}{27} + \frac{x^2}{36}\right)\Big|_1^3 \cong 2.07$

$$E(X^2) = \frac{125}{27}; \quad \sigma_X \cong 0.57.$$

$$P(2 < \overline{X} < 2.5) = P\left(2 < \frac{1}{24}\sum_{i=1}^{24} X_i < 2.5\right) = P\left(48 < \sum X_i < 60\right)$$

$$\approx -\Phi\left(\frac{48 - 24 \cdot 2.07}{0.57\sqrt{24}}\right) + \Phi\left(\frac{60 - 24(2.07)}{0.57\sqrt{24}}\right) \approx 0.726$$

Problem 6 Use the Chebyshev inequality:

$$P(X \geq 40) \leq P(|X - EX| \geq 40) = P(|X| \geq 40) \leq \frac{Var(X)}{40^2} = \frac{1}{160} = \underline{0.00625}$$

ENEE324 Spring 2016, HW9 Solutions

Problem 1. $N(t)$ is the number of arrivals in the interval $[0,t]$; $u<t$

(a)

$$P(N(u)=i \mid N(t)=n) = \frac{P(N(u)=i, N(t)=n)}{P(N(t)=n)} = \frac{P(N(u)=i, N(t)-N(u)=n-i)}{P(N(t)=n)}$$

The events in the numerator are independent

Moreover, by the memoryless property the difference $N(t)-N(u)$ has the same distribution as $N(t-u)$. We continue as follows:

$$= \frac{P(N(u)=i)\,P(N(t-u)=n-i)}{P(N(t)=n)}$$

$$= \frac{n!}{i!(n-i)!} \frac{e^{-\lambda u}(\lambda u)^i e^{-\lambda(t-u)}(\lambda(t-u))^{n-i}}{e^{-\lambda t}(\lambda t)^n} = \binom{n}{i} \frac{u^i (t-u)^{n-i}}{t^n}$$

$$= \binom{n}{i}\left(\frac{u}{t}\right)^i\left(1-\frac{u}{t}\right)^{n-i} \sim \text{Binom}\left(n, \frac{u}{t}\right)$$

(b) $$E[X_k \mid N(t)=n] = \int_0^t \frac{n!}{(n-k)!(k-1)!} \frac{x}{t}\left(\frac{x}{t}\right)^{k-1}\left(1-\frac{x}{t}\right)^{n-k} dx = \frac{n!\,t}{(n-k)!(k-1)!}\int_0^1 z^k (1-z)^{n-k}\,dz$$

$$= \frac{n!}{(n-k)!(k-1)!}\frac{k!(n-k)!}{(n+1)!}t = \frac{kt}{n+1}$$

Problem 2 $N(5)=8$. We need to find $P(X_1 \leq 2 \mid N(5)=8)$

~~Find~~

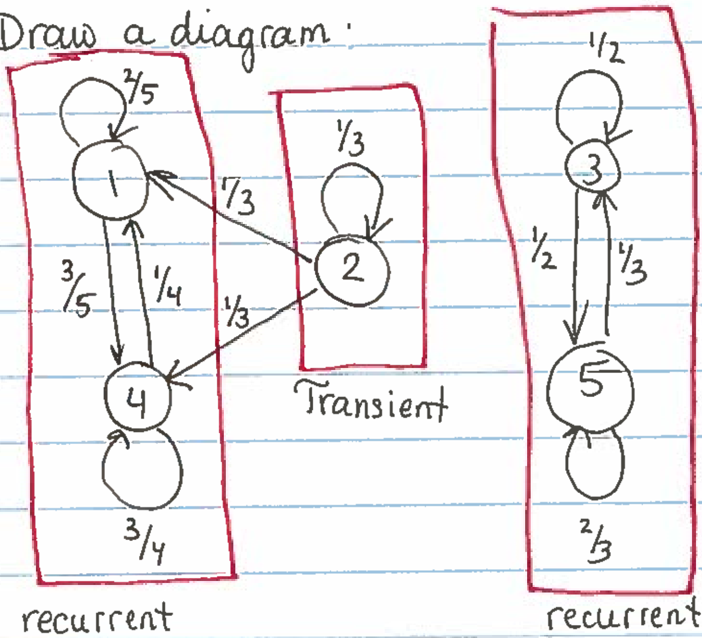
~~$P(X_1 \leq 2) = 1 - P(X_1 > 2) = 1 -$~~

From Prob 1 we see that $\quad n=8\,;\,t=5$

$$f_{X_1 \mid N(t)}(x \mid n) = \frac{n}{t}\left(\frac{x}{t}\right)^0\left(1-\frac{x}{t}\right)^{n-1} = \frac{8}{5}\left(1-\frac{x}{5}\right)^8, \quad 0 \leq x \leq 5$$

Thus $\displaystyle P(X_1 \leq 2) = \frac{8}{5}\int_0^2 \left(1-\frac{x}{5}\right)^8 dx = 1-\left(\frac{3}{5}\right)^8 \approx 0.984$

Problem 3 Draw a diagram:



Ans: $\{1,4\}$; $\{3,5\}$ are recurrent

$\{2\}$ transient

Problem 4. Let 0 denote a fall, 1 a clean run.

There are 4 possible states:

$$A=00;\ B=01;\ C=10;\ D=11$$

The transition matrix has the form

$$P=\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 7/8 & 1/8 & 0 & 0 \\ B & 0 & 0 & 1/2 & 1/2 \\ C & 1/2 & 1/2 & 0 & 0 \\ D & 0 & 0 & 1/8 & 7/8 \end{array}$$

$P(X_4=1 \mid X_1=X_2=1)$

(a) We need ~~$P(X_4=D \mid X_2=D)$ = prob. of returning from D to D in 2 steps. If the chain leaves D (i.e., no fall in the 3rd run), then it cannot return to D in one step because $p_{CD}=0$.~~

~~Thus $P(X_4=D \mid X_2=D)=(p_{DD})^2=\frac{49}{64}$~~

This can happen either by staying in state D, or by leaving D to C and then moving to B. Answer: $(p_{DD})^2+p_{DC}\,p_{CB}=53/64$.

(b) To answer this we need to find a steady-state distribution. Let π_i, $i\in\{A,B,C,D\}$ be the stationary probabilities of the chain. We have

$$\pi P=\pi$$

or

$$\frac{7}{8}\pi_A+\frac{1}{2}\pi_C=\pi_A$$

$$\frac{1}{8}\pi_A+\frac{1}{2}\pi_C=\pi_B$$

$$\frac{1}{2}\pi_B+\frac{1}{8}\pi_D=\pi_C$$

$$\frac{1}{2}\pi_B+\frac{7}{8}\pi_D=\pi_D$$

Exspress π_B, π_C, π_D via π_A: We obtain

$\pi_C=\frac{1}{4}\pi_A$ (from eq. 1)

$\pi_B=\frac{1}{4}\pi_A$ (from eq. 2)

$\pi_D=\pi_A$ (from eq. 4)

Using $\pi_A+\pi_B+\pi_C+\pi_D=1$, we find $\pi_A=\frac{2}{5}$, and so $\boxed{\pi_D=\frac{2}{5}}$

Problem 5 The transition matrix of the chain is:

$$P = \begin{bmatrix} 9/13 & 1/13 & 3/13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad P^2 = \begin{bmatrix} (9/13)^2 & \boxed{(9/13 \cdot 1/13) + 1/13} & (9/13)(3/13) + 3/13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) $p_{12} = \frac{1}{13}$;

$$r_{12}(2) = \frac{9}{13} \cdot \frac{1}{13} + \frac{1}{13}$$

$$r_{12}(3) = \left(\frac{9}{13}\right)^2\left(\frac{1}{13}\right) + \left(\frac{9}{13}\right)\frac{1}{13} + \frac{1}{13}$$

Generally, $$\underset{n \geq 2}{r_{12}(n)} = \frac{1}{13}\left[\left(\frac{9}{13}\right)^{n-1} + \left(\frac{9}{13}\right)^{n-2} + \dots + 1\right] = \frac{1}{13} \frac{(9/13)^n - 1}{-4/13}$$

$$\lim_{n \to \infty} r_{12}(n) = \frac{1}{4}$$

(b) Using the results under "Mean 1st passage time" pp. 367–368, we obtain the following. Before the start of the experiment, we may assume that the chain is in state ~~0~~ 1. We are interested in expected time till 1st hit of state 2:

$$t_{12} = 1 + t_{12} p_{12}$$

$$\Rightarrow t_{12} = 1 + \frac{1}{13} t_{12}$$

or $t_{12} = \frac{13}{12}$