

HW 1 - Solutions

35-39

- ④ a) $E = \{\text{First toss should result in a H}\}$
 b) $E = \{\text{All three tosses should be of the same type}\}$
 c) $E = \{\text{Exactly two tosses should be H}\}$
 d) $E = \{\text{At least one toss should give a T}\}$

⑭ a) $m_Y(y) = \begin{cases} 1/5 & ; y=2 \\ 1/5 & ; y=3 \\ 2/5 & ; y=4 \\ 1/5 & ; y=5 \\ 0 & ; \text{otherwise} \end{cases}$ or $m_Y(2)=1/5, m_Y(3)=1/5, m_Y(4)=2/5, m_Y(5)=1/5$

b) $m_Z(z) = \begin{cases} 1/5 & ; z=0 \\ 3/5 & ; z=1 \\ 1/5 & ; z=4 \\ 0 & ; \text{otherwise} \end{cases} \leftarrow (z=1, \text{ when both } x=-1 \text{ and } x=1)$

⑮ a) $P(\underbrace{A_1 \cup A_2}_{E_1}) = P(A_1) + P(A_2) - \underbrace{P(A_1 \cap A_2)}_{\geq 0}$

$\therefore P(E_1) = P(A_1 \cup A_2) \leq P(A_1) + P(A_2) \quad \text{--- ①}$

$P(A_1 \cup A_2 \cup A_3) = P(E_1 \cup A_3)$

$= P(E_1) + P(A_3) - \underbrace{P(E_1 \cap A_3)}_{\geq 0}$

from ① ↓

$= P(A_1) + P(A_2) + P(A_3) - P(E_1 \cap A_3)$

$\therefore P(A_1 \cup A_2 \cup A_3) \leq P(A_1) + P(A_2) + P(A_3)$

Extending it to n subsets $\Rightarrow P(A_1 \cup \dots \cup A_n) \leq P(A_1) + \dots + P(A_n)$

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) \leq 1$

$\therefore P(A) + P(B) - P(A \cap B) \leq 1$

$P(A \cap B) \geq P(A) + P(B) - 1$

① a) 2^n b) $A = \{\text{number of H's is even}\}$ if n is even,

$$|A| = \sum_{k=0}^{n/2} \binom{n}{2k}$$

$$= \binom{n}{0} + \binom{n}{2} + \dots + \binom{n}{n-2} + \binom{n}{n}$$

$$= \binom{n}{0} + \underbrace{\binom{n-1}{1} + \binom{n-1}{2} + \dots + \binom{n-1}{n-3} + \binom{n-1}{n-2}}_{2^{n-1} - \binom{n-1}{0} - \binom{n-1}{n-1}} + \binom{n}{n}$$

$$= \binom{n}{0} + 2^{n-1} - \binom{n-1}{0} - \binom{n-1}{n-1} + \binom{n}{n}$$

$$= 2^{n-1}$$

if n is odd,

$$|A| = \sum_{k=0}^{(n-1)/2} \binom{n}{2k+1}$$

$$= \binom{n}{1} + \binom{n}{3} + \dots + \binom{n}{n-1}$$

$$= \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \dots + \binom{n-1}{n-2} + \binom{n-1}{n-1}$$

$$= 2^{n-1}$$

$$\therefore |A| = \underline{\underline{2^{n-1}}}$$

c) $B_m = \{\text{exactly } m \text{ outcomes are T}\}$

$$|B_m| = \binom{n}{m} //$$

$$B = \bigcup_{m=1}^k B_m = B_1 \cup B_2 \cup \dots \cup B_k \Rightarrow B_m \text{'s are mutually exclusive}$$

$$|B| = \sum_{m=1}^k |B_m| = \sum_{m=1}^k \binom{n}{m} //$$

d) $C_i = \{\text{1st } k \text{ tosses } \Rightarrow i \text{ T's, next } n-k \text{ tosses } \Rightarrow m-i \text{ T's}\}$

$$|C_i| = \binom{k}{i} \binom{n-k}{m-i} //$$

$$C = \bigcup_{i=0}^k C_i = C_0 \cup C_1 \cup \dots \cup C_k \Rightarrow C_i \text{'s are mutually exclusive}$$

$$|C| = \sum_{i=0}^k |C_i| = \sum_{i=0}^k \binom{k}{i} \binom{n-k}{m-i} //$$

$$\textcircled{6} \quad P(\text{Charles wins}) = \sum_{i=7}^{10} \binom{10}{i} (0.75)^i (0.25)^{10-i} //$$

$$\begin{aligned} P(\text{Ruth wins}) &= P(\text{Charles guesses only 6 or less correctly}) \\ &= \sum_{i=0}^6 \binom{10}{i} (0.5)^i (0.5)^{10-i} \\ &= 0.5^{10} \sum_{i=0}^6 \binom{10}{i} // \end{aligned}$$

$$\begin{aligned} \textcircled{20} \quad \text{a) } P(\text{All 6 are hearts}) &= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} \cdot \frac{8}{47} \\ &= 8.43 \times 10^{-5} // \end{aligned}$$

$$\begin{aligned} \text{b) } P(3 \text{ A's, } 2 \text{ K's and } 1 \text{ Q}) &= \frac{\binom{4}{3} \cdot \binom{4}{2} \cdot \binom{4}{1}}{\binom{52}{6}} \\ &= 4.72 \times 10^{-6} // \end{aligned}$$

$$\begin{aligned} \text{c) } P(3 \text{ of one suit and } 3 \text{ of another suit}) &= \frac{\binom{4}{2} \binom{13}{3} \binom{13}{3}}{\binom{52}{6}} \\ &= 0.024 \\ &= \underline{\underline{\quad}} \end{aligned}$$

26) a) $P(HT) = p(1-p)$

$P(TH) = (1-p)p$

$P(HH) = p^2$

$P(TT) = (1-p)^2$



Tosses before (HT)

all pairs should be HH/TT



Tosses before TH

all pairs should be HH/TT

$A = \{ \text{getting HH or TT at a given pair of tosses} \}$

$P(A) = p^2 + (1-p)^2$

$P(\text{win}) = \sum_{n=0}^{\infty} (P(A))^n P(HT)$

$= \sum_{n=0}^{\infty} (p^2 + (1-p)^2)^n p(1-p)$

$= p(1-p) \frac{1}{1 - (p^2 + (1-p)^2)}$

$= \frac{p(1-p)}{1 - (p^2 + 1 + p^2 - 2p)}$

$= \frac{p(1-p)}{2p - 2p^2}$

$= \frac{p(1-p)}{2p(1-p)}$

$= \frac{1}{2}$

② 5 - red 12 - blue 9 - white with 5 different permutations

$$\begin{aligned}
 a) P(\text{At least 4 blue balls}) &= P(BBBBB) + P(BBBBBR) + P(BBBBBW) \\
 &= \left(\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22} \right) + \left(\frac{12 \times 11 \times 10 \times 9 \times 5}{26 \times 25 \times 24 \times 23 \times 22} \times 5 \right) \\
 &\quad + \left(\frac{12 \times 11 \times 10 \times 9 \times 9}{26 \times 25 \times 24 \times 23 \times 22} \times 5 \right) \\
 &= \underline{\underline{0.12}}
 \end{aligned}$$

$$\begin{aligned}
 b) P(5 \text{ balls of same color}) &= P(RRRRR) + P(BBBBB) + P(WWWWW) \\
 &= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22} + \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22} \\
 &\quad + \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22} \\
 &= \underline{\underline{0.014}}
 \end{aligned}$$

$$\begin{aligned}
 c) \cancel{P(\text{At most 3 white balls})} &= \cancel{P(\text{No W}) + P(1W) + P(2W) + P(3W)} \\
 &\quad \cancel{\frac{17 \cdot 16 \cdot 15 \cdot 14}{26 \cdot 26 \cdot 26 \cdot 26}}
 \end{aligned}$$

$$\begin{aligned}
 c) P(\text{At most 3 white balls}) &= 1 - P(4W) - P(5W) \\
 &= 1 - P(WWWW R) - P(WWWW B) - P(WWWW W) \\
 &\quad \quad \quad \nwarrow \quad \quad \nearrow \\
 &\quad \quad \quad \text{with 5 different permutations} \\
 &= 1 - \left\{ \left(\frac{9 \times 8 \times 7 \times 6 \times 5}{26 \times 25 \times 24 \times 23 \times 22} \right) + \left(\frac{9 \times 8 \times 7 \times 6 \times 12}{26 \times 25 \times 24 \times 23 \times 22} \right) \right\} \times 5 \\
 &\quad \quad \quad - \left(\frac{9 \times 8 \times 7 \times 6 \times 5}{26 \times 25 \times 24 \times 23 \times 22} \right) \\
 &= \underline{\underline{0.966}}
 \end{aligned}$$

HW 3

- ① 9 - Mathematicians 12 - Engineers
need - 4 mathematicians, 5 engineers.

$$\begin{aligned} \text{a) Any math. any eng.} &= \binom{9}{4} \times \binom{12}{5} \\ &= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2} \times \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2} \\ &= \underline{\underline{99792}} \end{aligned}$$

$$\begin{aligned} \text{b) with one particular eng.} &= \binom{11}{4} \times \binom{9}{4} \\ &= \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2} \times \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2} \\ &= \underline{\underline{41580}} \end{aligned}$$

$$\begin{aligned} \text{c) without 2 particular math.} &= \binom{7}{4} \times \binom{12}{5} \\ &= \frac{7 \times 6 \times 5}{3 \times 2} \times \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2} \\ &= \underline{\underline{27720}} \end{aligned}$$

- ② $n \rightarrow m$ cars
 $\quad \quad \rightarrow n-m$ trucks

$$\# \text{ different orderings} = \frac{n!}{m!(n-m)!} \leftarrow \binom{n}{m} \text{ number of ways to position the indistinguishable cars}$$

because there are only two types of vehicles, trucks are automatically placed in all possible different ways, within the $\binom{n}{m}$ ways.

Problem 5. We have

$$\checkmark P(A_i) = 1 - P(A_i^c) = 1 - (1 - p_i)^k, \quad i = 1, 2, 3$$

$$\checkmark P(A_i \cup A_j) = 1 - P((A_i \cup A_j)^c) = 1 - P(\text{each of the coupons is of type } i \neq j \text{ other than } i \text{ or } j) \\ = 1 - (1 - p_i - p_j)^k$$

$$\checkmark P(A_i | A_j) = \frac{P(A_i \cap A_j)}{P(A_j)}$$

For the numerator, use $P(A_i \cap A_j) = P(A_i) + P(A_j) - P(A_i \cup A_j)$

$$= 1 - (1 - p_i)^k + 1 - (1 - p_j)^k - 1 + (1 - p_i - p_j)^k \\ = 1 - (1 - p_i)^k - (1 - p_j)^k + (1 - p_i - p_j)^k$$

Answer:

$$P(A_i | A_j) = \frac{1 - (1 - p_i)^k - (1 - p_j)^k + (1 - p_i - p_j)^k}{1 - (1 - p_i)^k}$$

Problem 3. Consider subsets of size k of the set of numbers $\{1, 2, \dots, n\}$.

If the largest element of the subset is i , then this subset can be chosen in $\binom{i-1}{k-1}$ ways, $i \geq k$. Going over all possible choices of $i = k, k+1, \dots, n$, we will enumerate all the k -subsets of $\{1, 2, \dots, n\}$. Thus

$$\sum_{i=k}^n \binom{i-1}{k-1} = \binom{n}{k}$$

2nd solution: From the Pascal Triangle we know that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

We have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \xrightarrow{\text{use the boxed equation for this term}} \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \binom{n-3}{k} \xleftarrow{\text{and for this}} \\ \xleftarrow{\text{and for this}} \dots = \sum_{i=k}^n \binom{n-i}{k-1}$$

Problem 4. Let L be the event that the test did not detect outcome 1.

Let $A_i, i=1,2,3$ be the event that the outcome of the experiment is i .

By the total probability formula

$$P(L) = \alpha_1 S_1 + 1 \cdot S_2 + 1 \cdot S_3 = \alpha_1 S_1 + S_2 + S_3$$

$$p_1 = P(A_1|L) = \frac{P(LA_1)}{P(L)} = \frac{P(L|A_1)P(A_1)}{P(L)} = \frac{\alpha_1 S_1}{P(L)}$$

$$p_2 = P(A_2|L) = \frac{P(LA_2)}{P(L)} = \frac{P(L|A_2)P(A_2)}{P(L)} = \frac{S_2}{P(L)}$$

$$p_3 = \frac{S_3}{P(L)} \quad \text{in exactly the same way as } p_2$$

application of
the Bayes
formula

⑥ N : number of onsets of the disease within 2 years

$$P(+ \text{ effect} | N=3) = \frac{P(+ \text{ effect}, N=3)}{P(N=3)}$$

$$= \frac{P(N=3 | + \text{ effect}) P(+ \text{ effect})}{P(N=3 | + \text{ effect}) P(+ \text{ effect}) + P(N=3 | \text{no effect}) P(\text{no effect})}$$

$$= \frac{e^{-6} (6)^3 \times 0.6}{e^{-6} (6)^3 \times 0.6 + e^{-8} (8)^3 \times 0.4}$$

$$= \frac{e^{-6} (6)^3 \times 0.6}{e^{-6} (6)^3 \times 0.6 + e^{-8} (8)^3 \times 0.4}$$

$$= \frac{e^{-6} (6)^3 \times 0.6}{e^{-6} (6)^3 \times 0.6 + e^{-8} (8)^3 \times 0.4}$$

$$= \frac{e^{-6} (6)^3 \times 0.6}{e^{-6} (6)^3 \times 0.6 + e^{-8} (8)^3 \times 0.4}$$

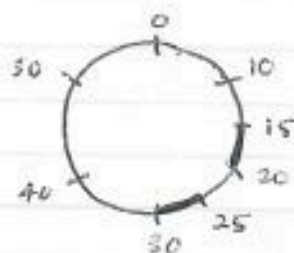
$$= \frac{0.6}{0.6 + 0.4 e^{-2} \left(\frac{4}{3}\right)^3}$$

$$= \frac{0.6}{0.6 + 0.4 e^{-2} \left(\frac{4}{3}\right)^3}$$

$$= \frac{0.6}{0.6 + 0.4 e^{-2} \left(\frac{4}{3}\right)^3}$$

$$= 0.324$$

⑦

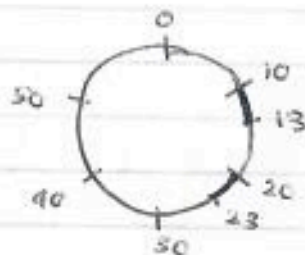


a) $P(\text{wait} < 5) = P(15 < \text{arrival} < 20) + P(25 < \text{arrival} < 30)$

$$= \frac{10}{20}$$

$$= 0.5$$

b)



$P(\text{wait} > 7) = P(10 < \text{arrival} < 13) + P(20 < \text{arrival} < 23)$

$$= \frac{3}{20}$$

$$= 0.3$$

HW 4

$$\textcircled{1} \quad a) \quad \int_{-\infty}^{\infty} f_X(x) dx = 1$$

$$\int_{-1}^1 c(1-x^2)+1 \, dx = 1$$

$$(c+1)[x]_{-1}^1 - c \int_{-1}^1 x^2 dx = 1$$

$$2(c+1) - c \left[\frac{x^3}{3} \right]_{-1}^1 = 1$$

$$2(c+1) - \frac{c}{3}(2) = 1$$

$$\frac{4}{3}c = -1$$

$$c = -\frac{3}{4}$$

$$b) \quad F_X(x) = \int_{-\infty}^x f_X(x) dx = \int_{-1}^x -\frac{3}{4}(1-x^2)+1 \, dx$$

$$= \frac{1}{4}[x]_{-1}^x + \frac{3}{4} \int_{-1}^x x^2 dx$$

$$= \frac{1}{4}(x+1) + \frac{3}{4} \left[\frac{x^3}{3} \right]_{-1}^x$$

$$= \frac{1}{4}(x+1) + \frac{1}{4}(x^3+1)$$

$$F_X(x) = \begin{cases} 0 & ; x \leq -1 \\ \frac{1}{4}(x+x^3+2) & ; -1 \leq x < 1 \\ 1 & ; x \geq 1 \end{cases}$$

$$c) \quad EX = \int_{-\infty}^{\infty} x(c(1-x^2)+1) dx = \int_{-1}^1 \left(-\frac{3}{4}x^3 + \frac{1}{4}x \right) dx$$

$$= \frac{3}{4} \left[\frac{x^4}{4} \right]_{-1}^1 + \frac{1}{4} \left[\frac{x^2}{2} \right]_{-1}^1$$

$$= 0$$

$$\# \quad \text{Var}(X) = E(X^2) - (EX)^2 = EX^2$$

$$EX^2 = \int_{-1}^1 x^2 \left[-\frac{3}{4}(1-x^2)+1 \right] dx = \int_{-1}^1 \left(\frac{3}{4}x^4 + \frac{1}{4}x^2 \right) dx$$

$$= \frac{3}{4} \left[\frac{x^5}{5} \right]_{-1}^1 + \frac{1}{4} \left[\frac{x^3}{3} \right]_{-1}^1$$

$$= \frac{3}{10} + \frac{1}{6}$$

$$= \frac{14}{30} = \frac{7}{15}$$

$$(5) X \sim \exp(\lambda) \quad \frac{1}{\lambda} = 10 \Rightarrow \lambda = 0.1$$

$$a) P(\text{exactly 5 mins}) = 0$$

$$P(\text{lasts 5 mins}) = P(X \geq 5)$$

$$= \int_5^{\infty} 0.1 e^{-0.1x} dx$$

$$= 0.1 \left[\frac{e^{-0.1x}}{-0.1} \right]_5^{\infty}$$

$$= -(0 - e^{-0.5})$$

$$= \underline{\underline{e^{-0.5}}}$$

$$b) P(X \geq t+4 | X \geq t) = \frac{P(X \geq t+4, X \geq t)}{P(X \geq t)}$$

$$= \frac{P(X \geq t+4)}{P(X \geq t)}$$

$$= \frac{\int_{t+4}^{\infty} 0.1 e^{-0.1x} dx}{\int_t^{\infty} 0.1 e^{-0.1x} dx}$$

$$= \frac{0.1 \left[\frac{e^{-0.1x}}{-0.1} \right]_{t+4}^{\infty}}{0.1 \left[\frac{e^{-0.1x}}{-0.1} \right]_t^{\infty}}$$

$$= \frac{-(0 - e^{-0.1(t+4)})}{-(0 - e^{-0.1t})}$$

$$= \frac{e^{-0.1(t+4)}}{e^{-0.1t}}$$

$$= \underline{\underline{e^{-0.4}}}$$

$$(6) U = \min(X, Y)$$

$$P(U \leq x) = 1 - P(U > x)$$

$$= 1 - P(X > x, Y > x)$$

$$= 1 - P(X > x) P(Y > x)$$

$$= 1 - (1 - P(X \leq x))(1 - P(Y \leq x))$$

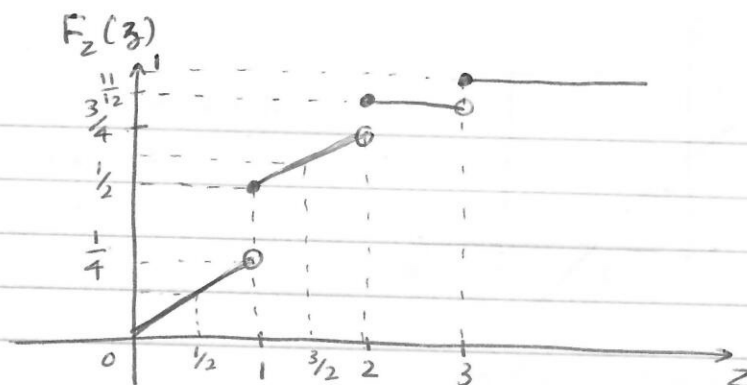
$$= 1 - (1 - F_X(x))(1 - F_Y(x)) //$$

$$V = \max(X, Y)$$

$$P(V \leq x) = P(X \leq x, Y \leq x) = P(X \leq x) P(Y \leq x)$$

$$= F_X(x) F_Y(x) //$$

7



a)

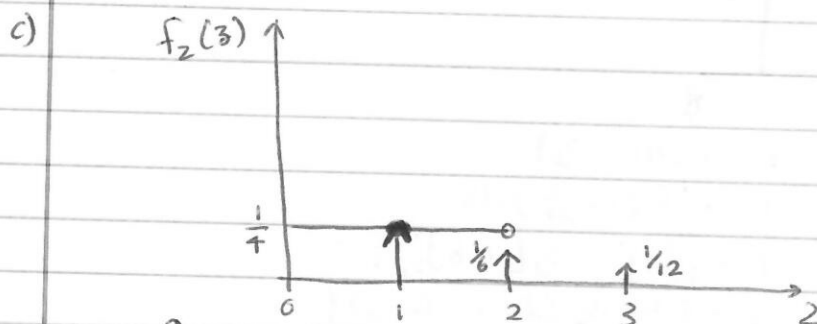
$$P(X=1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$P(X=2) = \frac{11}{12} - \frac{3}{4} = \frac{1}{6}$$

$$P(X=3) = 1 - \frac{11}{12} = \frac{1}{12}$$

b)

$$P(\frac{1}{2} < X < \frac{3}{2}) = F_2(\frac{3}{2}) - F_2(\frac{1}{2}) = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$$



$$EX = \int_0^2 \frac{1}{4} dx + \frac{1}{6} \times 2 + \frac{1}{12} \times 3 + \frac{1}{4} \times 1$$

$$= \frac{1}{4} \left[\frac{x^2}{2} \right]_0^2 + \frac{1}{3} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{13}{12} = \frac{4}{3}$$

$$EX^2 = \int_0^2 \frac{1}{4} x^2 dx + 1 \times \frac{1}{4} + 4 \times \frac{1}{6} + 9 \times \frac{1}{12}$$

$$= \frac{1}{4} \left[\frac{x^3}{3} \right]_0^2 + \frac{1}{4} + \frac{2}{3} + \frac{3}{4}$$

$$= \frac{1}{4} \left(\frac{8}{3} \right) + \frac{5}{3}$$

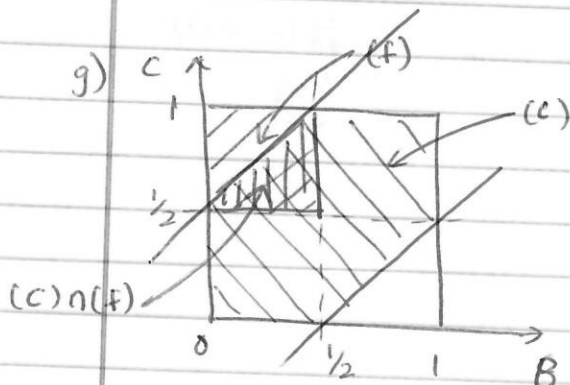
$$= \frac{2}{3} + \frac{5}{3}$$

$$= \frac{7}{3}$$

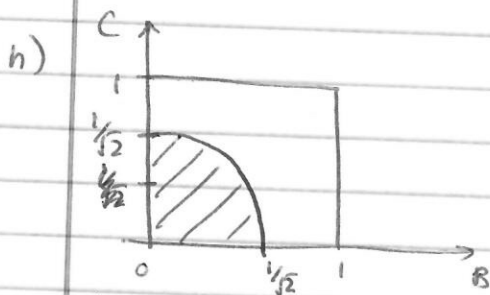
$$\begin{aligned}
 d) P(\max(B, C) < 1/2) &= P(B < 1/2, C < 1/2) \\
 &= P(B < 1/2) P(C < 1/2) \\
 &= 1/2 \times 1/2 \\
 &= \underline{1/4}
 \end{aligned}$$

$$\begin{aligned}
 e) P(\min\{B, C\} < 1/2) &= 1 - P(\min\{B, C\} \geq 1/2) \\
 &= 1 - P(B \geq 1/2, C \geq 1/2) \\
 &= 1 - P(B \geq 1/2) P(C \geq 1/2) \\
 &= 1 - 1/2 \cdot 1/2 \\
 &= \underline{3/4}
 \end{aligned}$$

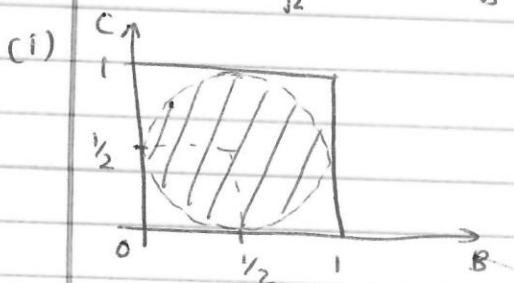
$$\begin{aligned}
 f) P(B < 1/2, 1 - C < 1/2) &= P(B < 1/2, C > 1/2) \\
 &= P(B < 1/2) P(C > 1/2) \\
 &= 1/2 \cdot 1/2 \\
 &= \underline{1/4}
 \end{aligned}$$



$$P((c) \cap (f)) = \underline{1/8}$$



$$P(B^2 + C^2 \leq 1/2) = \frac{\pi}{2} \times \frac{1}{4} = \underline{\frac{\pi}{8}}$$



$$P((B - 1/2)^2 + (C - 1/2)^2 < 1/4) = \underline{\frac{\pi}{4}}$$

HW 4

① a) $\int_{-\infty}^{\infty} f_X(x) dx = 1$

$$\int_{-1}^1 c(1-x^2)+1 dx = 1$$

$$(c+1)[x]_{-1}^1 - c \int_{-1}^1 x^2 dx = 1$$

$$2(c+1) - c \left[\frac{x^3}{3} \right]_{-1}^1 = 1$$

$$2(c+1) - \frac{c}{3}(2) = 1$$

$$\frac{4}{3}c = -1$$

$$c = \underline{\underline{-\frac{3}{4}}}$$

b) $F_X(x) = \int_{-\infty}^x f_X(x) dx = \int_{-1}^x -\frac{3}{4}(1-x^2)+1 dx$

$$= \frac{1}{4}[x]_{-1}^x + \frac{3}{4} \int_{-1}^x x^2 dx$$

$$= \frac{1}{4}(x+1) + \frac{3}{4} \left[\frac{x^3}{3} \right]_{-1}^x$$

$$= \frac{1}{4}(x+1) + \frac{1}{4}(x^3+1)$$

$$F_X(x) = \begin{cases} 0 & ; x \leq -1 \\ \frac{1}{4}(x+x^3+2) & ; -1 \leq x < 1 \\ 1 & ; x \geq 1 \end{cases}$$

c) $EX = \int_{-\infty}^{\infty} x(c(1-x^2)+1) dx = \int_{-1}^1 +\frac{3}{4}x^3 + \frac{1}{4}x dx$

$$= \frac{3}{4} \left[\frac{x^4}{4} \right]_{-1}^1 + \frac{1}{4} \left[\frac{x^2}{2} \right]_{-1}^1$$

$$= 0$$

* $\text{Var}(X) = E(X^2) - (EX)^2 = EX^2$

$$EX^2 = \int_{-1}^1 x^2 \left[-\frac{3}{4}(1-x^2)+1 \right] dx = \int_{-1}^1 \frac{3}{4}x^4 + \frac{1}{4}x^2 dx$$

$$= \frac{3}{4} \left[\frac{x^5}{5} \right]_{-1}^1 + \frac{1}{4} \left[\frac{x^3}{3} \right]_{-1}^1$$

$$= \frac{3}{10} + \frac{1}{6}$$

$$= \frac{14}{30} = \underline{\underline{\frac{7}{15}}}$$

$$(5) X \sim \exp(\lambda) \quad \frac{1}{\lambda} = 10 \Rightarrow \lambda = 0.1$$

$$a) P(\text{exactly 5 mins}) = 0$$

$$P(\text{lasts 5 mins}) = P(X \geq 5)$$

$$= \int_5^{\infty} 0.1 e^{-0.1x} dx$$

$$= 0.1 \left[\frac{e^{-0.1x}}{-0.1} \right]_5^{\infty}$$

$$= -(0 - e^{-0.5})$$

$$= e^{-0.5}$$

$$b) P(X \geq t+4 | X \geq t) = \frac{P(X \geq t+4, X \geq t)}{P(X \geq t)}$$

$$= \frac{P(X \geq t+4)}{P(X \geq t)}$$

$$= \frac{\int_{t+4}^{\infty} 0.1 e^{-0.1x} dx}{\int_t^{\infty} 0.1 e^{-0.1x} dx}$$

$$= \frac{0.1 \left[\frac{e^{-0.1x}}{-0.1} \right]_{t+4}^{\infty}}{0.1 \left[\frac{e^{-0.1x}}{-0.1} \right]_t^{\infty}}$$

$$= \frac{-(0 - e^{-0.1(t+4)})}{-(0 - e^{-0.1t})}$$

$$= \frac{e^{-0.1(t+4)}}{e^{-0.1t}}$$

$$= e^{-0.1}$$

$$= e^{-0.1}$$

$$(6) U = \min(X, Y)$$

$$P(U \leq x) = 1 - P(U > x)$$

$$= 1 - P(X > x, Y > x)$$

$$= 1 - P(X > x) P(Y > x)$$

$$= 1 - (1 - P(X \leq x))(1 - P(Y \leq x))$$

$$= 1 - (1 - F_x(x))(1 - F_y(x)) //$$

$$V = \max(X, Y)$$

$$P(V \leq x) = P(X \leq x, Y \leq x) = P(X \leq x) P(Y \leq x)$$

$$= F_x(x) F_y(x) //$$

$$\begin{aligned}
 F_u(x) &= 1 - (1 - [1 - e^{-x}]) (1 - [1 - e^{-x}]) \\
 &= 1 - e^{-x} \cdot e^{-x} \\
 &= 1 - e^{-2x}
 \end{aligned}$$

$$\# \quad u \sim \exp(2)$$

$$\begin{aligned}
 F_v(x) &= (1 - e^{-x})(1 - e^{-x}) \\
 &= 1 + e^{-2x} - 2e^{-x}
 \end{aligned}$$

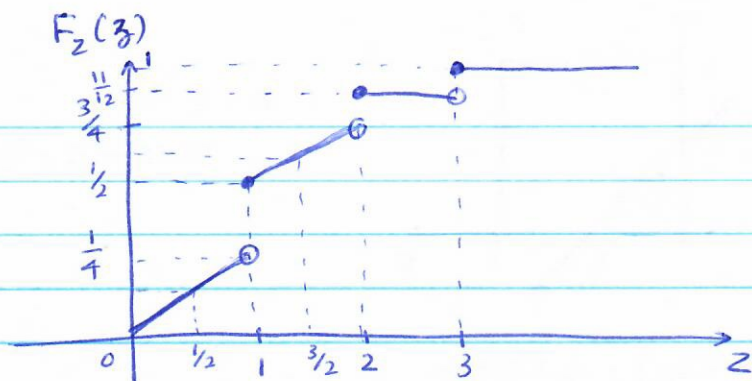
$$f_v(x) = -2e^{-2x} + 2e^{-x}$$

$$\begin{aligned}
 \text{mean}(v) &= \int_0^{\infty} x f_v(x) dx \\
 &= \int_0^{\infty} (-2e^{-2x} + 2e^{-x}) x dx \\
 &= -2 \int_0^{\infty} x e^{-2x} dx + 2 \int_0^{\infty} x e^{-x} dx \quad \begin{array}{l} u=x \\ \int dv = \int e^{-2x} \\ = \frac{e^{-2x}}{-2} \end{array} \\
 &= -2 \left\{ \left[-\frac{1}{2} x e^{-2x} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-2x}}{-2} dx \right\} \\
 &\quad + 2 \left\{ \left[-e^{-x} x \right]_0^{\infty} - \int_0^{\infty} -e^{-x} dx \right\} \\
 &= -2 \left\{ \frac{1}{2} \left[\frac{e^{-2x}}{-2} \right]_0^{\infty} \right\} + 2 \left\{ \left[\frac{e^{-x}}{-1} \right]_0^{\infty} \right\} \\
 &= \frac{1}{2} (0 - 1) - 2 (0 - 1) \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 E(V^2) &= \int_0^{\infty} x^2 f_v(x) dx \quad \begin{array}{l} u=x^2 \\ du=2x dx \\ \int dv = \int e^{-2x} \\ = \frac{e^{-2x}}{-2} \end{array} \\
 &= \int_0^{\infty} x^2 (-2e^{-2x} + 2e^{-x}) dx \\
 &= -2 \int_0^{\infty} x^2 e^{-2x} dx + 2 \int_0^{\infty} x^2 e^{-x} dx \\
 &= -2 \left\{ \left[-\frac{1}{2} x^2 e^{-2x} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-2x}}{-2} \cdot 2x dx \right\} \\
 &\quad + 2 \left\{ \left[-x^2 e^{-x} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-x}}{-1} \cdot 2x dx \right\} \\
 &= -2 \int_0^{\infty} x e^{-2x} dx + 2 \int_0^{\infty} 2x e^{-x} dx \\
 &= -2 \left\{ \left[-\frac{1}{2} e^{-2x} x \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-2x}}{-2} dx \right\} + 4 \left\{ \left[-x e^{-x} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-x}}{-1} dx \right\} \\
 &= -2 \left\{ \frac{1}{2} \left[\frac{e^{-2x}}{-2} \right]_0^{\infty} \right\} + 4 \left\{ \left[\frac{e^{-x}}{-1} \right]_0^{\infty} \right\} \\
 &= -2 \left\{ -\frac{1}{4} (0 - 1) \right\} - 4 \left\{ (0 - 1) \right\} \\
 &= -\frac{1}{2} + 4 \\
 &= \frac{7}{2}
 \end{aligned}$$

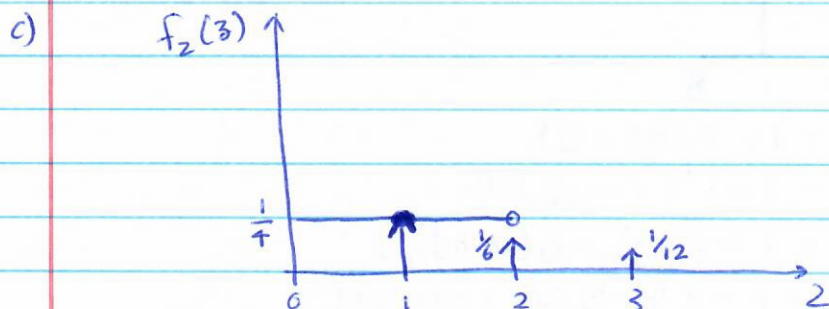
$$\begin{aligned}
 \text{Var}(v) &= E(V^2) - (E(v))^2 \\
 &= \frac{7}{2} - \frac{9}{4} \\
 &= \frac{5}{4}
 \end{aligned}$$

7



a) $P(X=1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$
 $P(X=2) = \frac{11}{12} - \frac{3}{4} = \frac{1}{6}$
 $P(X=3) = 1 - \frac{11}{12} = \frac{1}{12}$

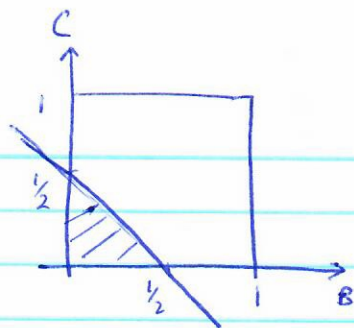
b) $P(\frac{1}{2} < X < \frac{3}{2}) = F_2(\frac{3}{2}) - F_2(\frac{1}{2}) = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$



$$\begin{aligned} EX &= \int_0^2 \frac{1}{4} dx + \frac{1}{6} \times 2 + \frac{1}{12} \times 3 + \frac{1}{4} \times 1 \\ &= \frac{1}{4} \left[\frac{x^2}{2} \right]_0^2 + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} \\ &= \frac{12}{12} + \frac{4}{12} + \frac{3}{12} + \frac{3}{12} \\ &= \frac{22}{12} = \frac{11}{6} \end{aligned}$$

$$\begin{aligned} EX^2 &= \int_0^2 \frac{1}{4} x^2 dx + 1 \times \frac{1}{4} + 4 \times \frac{1}{6} + 9 \times \frac{1}{12} \\ &= \frac{1}{4} \left[\frac{x^3}{3} \right]_0^2 + \frac{1}{4} + \frac{2}{3} + \frac{3}{4} \\ &= \frac{1}{4} \left(\frac{8}{3} \right) + \frac{5}{3} \\ &= \frac{2}{3} + \frac{5}{3} \\ &= \frac{7}{3} \end{aligned}$$

2) a)

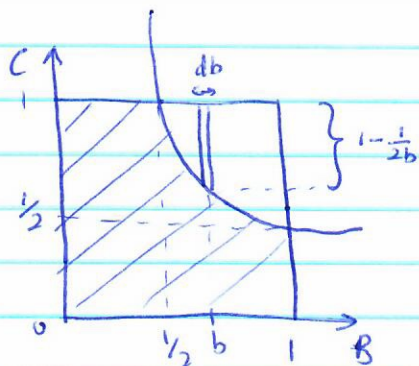


$$B + C < 1/2$$

$$C < -B + 1/2$$

$$P(B + C < 1/2) = \frac{1/2 \times 1/2 \times 1/2}{1 \times 1} = \frac{1}{8}$$

b)

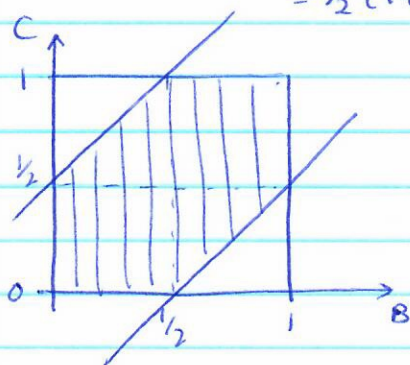


$$BC < 1/2$$

$$C < 1/2 B^{-1}$$

$$\begin{aligned} P(BC < 1/2) &= 1 - P(BC \geq 1/2) \\ &= 1 - \int_{1/2}^1 (1 - \frac{1}{2b}) db \\ &= 1 - \left\{ [b]_{1/2}^1 - \frac{1}{2} [\ln b]_{1/2}^1 \right\} \\ &= 1 - \left\{ 1/2 - \frac{1}{2} (\ln 1 - \ln 1/2) \right\} \\ &= 1/2 + \frac{1}{2} (\ln 2) \\ &= \frac{1}{2} (1 + \ln 2) \end{aligned}$$

c)



$$|B - C| < 1/2$$

$$-1/2 < B - C < 1/2$$

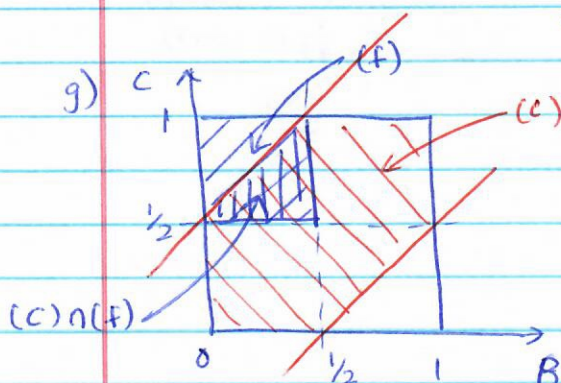
$$C < B + 1/2 \text{ and } C > B - 1/2$$

$$P(|B - C| < 1/2) = 1 - (\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 2) = 1 - \frac{1}{4} = \frac{3}{4}$$

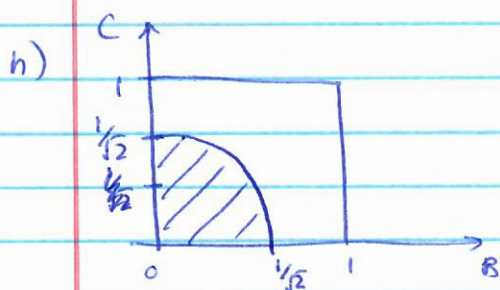
$$\begin{aligned}
 d) P(\max(B, C) < 1/2) &= P(B < 1/2, C < 1/2) \\
 &= P(B < 1/2) P(C < 1/2) \\
 &= 1/2 \times 1/2 \\
 &= 1/4 \\
 &=
 \end{aligned}$$

$$\begin{aligned}
 e) P(\min\{B, C\} < 1/2) &= 1 - P(\min\{B, C\} \geq 1/2) \\
 &= 1 - P(B \geq 1/2, C \geq 1/2) \\
 &= 1 - P(B \geq 1/2) P(C \geq 1/2) \\
 &= 1 - 1/2 \cdot 1/2 \\
 &= 3/4 \\
 &=
 \end{aligned}$$

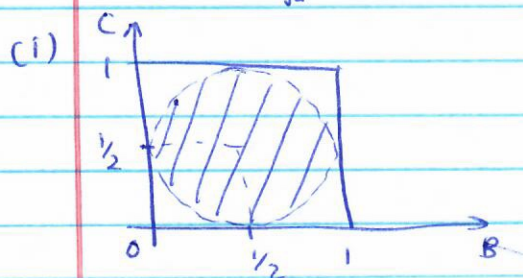
$$\begin{aligned}
 f) P(B < 1/2, 1 - C < 1/2) &= P(B < 1/2, C > 1/2) \\
 &= P(B < 1/2) P(C > 1/2) \\
 &= 1/2 \cdot 1/2 \\
 &= 1/4 \\
 &=
 \end{aligned}$$



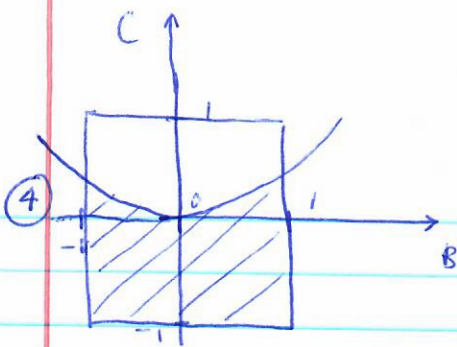
$$P((c) \cap (f)) = 1/8 =$$



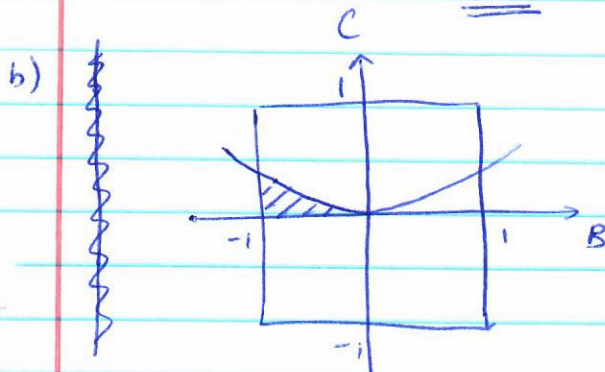
$$P(B^2 + C^2 \leq 1/2) = \frac{\pi}{2} \times \frac{1}{4} = \frac{\pi}{8} =$$



$$P((B - 1/2)^2 + (C - 1/2)^2 < 1/4) = \frac{\pi}{4} =$$



$$\begin{aligned}
 a) \quad P(B^2 - 4C \geq 0) &= \frac{1}{4} \left\{ 1 \times 2 + \int_{-1}^1 \frac{B^2}{4} dB \right\} \\
 &= \frac{1}{4} \left(2 + \frac{1}{4} \left[\frac{B^3}{3} \right]_{-1}^1 \right) \\
 &= \frac{1}{4} \left(2 + \frac{1}{2} (2) \right) \\
 &= \frac{1}{4} \left(\frac{13}{6} \right) \\
 &= \underline{\underline{13/24}}
 \end{aligned}$$



$$\begin{aligned}
 P(B^2 - 4C \geq 0, B \leq 0, C \geq 0) &= \frac{1}{4} \left\{ \int_{-1}^0 \frac{B^2}{4} dB \right\} / 1 \\
 &= \frac{1}{4} - \frac{1}{4} \left[\frac{B^3}{3} \right]_{-1}^0 \\
 &= \frac{1}{48} (0 - (-1)) \\
 &= \underline{\underline{1/48}}
 \end{aligned}$$

HW 5

pg 219 (2) ① $U \sim \text{uniform } [0, 1]$

a) $Y = \frac{1}{u+1}$

$$P(Y \leq y) = P\left(\frac{1}{u+1} \leq y\right)$$

$$= P(u+1 \geq \frac{1}{y})$$

$$= P(u \geq \frac{1}{y} - 1)$$

$$F_Y(y) = \begin{cases} 1 - (\frac{1}{y} - 1) & ; 0 \leq \frac{1}{y} - 1 \leq 1 \Rightarrow \frac{1}{2} \leq y \leq 1 \\ 0 & ; \frac{1}{y} - 1 \geq 1 \Rightarrow y \leq \frac{1}{2} \\ 1 & ; \frac{1}{y} - 1 \leq 0 \Rightarrow y \geq 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & ; y \leq \frac{1}{2} \\ 2 - \frac{1}{y} & ; \frac{1}{2} \leq y \leq 1 \\ 1 & ; y \geq 1 \end{cases} //$$

$$f_Y(y) = \begin{cases} 0 & ; y \leq \frac{1}{2} \\ \frac{1}{y^2} & ; \frac{1}{2} \leq y \leq 1 \\ 0 & ; y \geq 1 \end{cases} //$$

b) $Y = \log(u+1)$

$$P(Y \leq y) = P(\log(u+1) \leq y)$$

$$= P(u+1 \leq e^y)$$

$$= P(u \leq e^y - 1)$$

$$= \begin{cases} 0 & ; e^y - 1 \leq 0 \Rightarrow y \leq 0 \\ e^y - 1 & ; 0 \leq e^y - 1 \leq 1 \Rightarrow 0 \leq y \leq \ln(2) \\ 1 & ; e^y - 1 \geq 1 \Rightarrow y \geq \ln(2) \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & ; y \leq 0 \\ e^y - 1 & ; 0 \leq y \leq \ln(2) \\ 1 & ; y \geq \ln(2) \end{cases} //$$

$$f_Y(y) = \begin{cases} 0 & ; y \leq 0 \\ e^y & ; 0 \leq y \leq \ln(2) \\ 0 & ; y \geq \ln(2) \end{cases} //$$

P
221, 18

(2) $Y = X + b$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X + b \leq y) \\ &= P(X \leq y - b) \\ &= F_X(y - b) \\ &= \end{aligned}$$

$Z = aX$

$a > 0$ $F_Z(z) = P(Z \leq z) = P(aX \leq z) = P(X \leq z/a) = F_X(z/a)$

$a = 0$ $Z = 0$ $F_Z(z) = \begin{cases} 0 & ; z < 0 \\ 1 & ; z \geq 0 \end{cases}$

$a < 0$ $F_Z(z) = P(Z \leq z) = P(aX \leq z) = P(X \geq z/a) = 1 - P(X < z/a) \\ = 1 - P(X \leq z/a) \\ = 1 - F_X(z/a)^{\text{(continuous)}} \\ =$

$W = aX + b$

$a > 0$ $F_W(w) = P(W \leq w) = P(aX + b \leq w) = P(X \leq \frac{w-b}{a}) = F_X(\frac{w-b}{a})$

$a = 0$ $W = b$ $F_W(w) = \begin{cases} 0 & ; w < b \\ 1 & ; w \geq b \end{cases}$

$a < 0$ $F_W(w) = P(W \leq w) = P(aX + b \leq w) = P(X \geq \frac{w-b}{a}) = 1 - P(X < \frac{w-b}{a}) \\ = 1 - P(X \leq \frac{w-b}{a}) \\ = 1 - F_X(\frac{w-b}{a})^{\text{(continuous)}} \\ =$

P
223, 34

(3) $f(x) = \lambda e^{-\lambda x}$ $g(x) = \mu e^{-\mu x}$

$$\begin{aligned} P(X < Y) &= \int_0^{\infty} P(Y > x | X = x) f_X(x) dx \\ &= \int_0^{\infty} \int_x^{\infty} f_Y(y) dy f_X(x) dx \\ &= \int_0^{\infty} \int_x^{\infty} \lambda e^{-\lambda y} dy f_X(x) dx = \int_0^{\infty} \int_x^{\infty} g(y) dy f(x) dx \\ &= \int_0^{\infty} \lambda \left[\frac{e^{-\lambda y}}{-\lambda} \right]_x^{\infty} f_X(x) dx = \int_0^{\infty} (1 - e^{-\lambda x}) f_X(x) dx \\ &= \int_0^{\infty} (1 - G(x)) f(x) dx \\ &= \int_0^{\infty} \lambda e^{-\lambda x} \mu e^{-\mu x} dx = \int_0^{\infty} f(x) (1 - G(x)) dx \\ &= \mu \int_0^{\infty} e^{-(\lambda+\mu)x} dx \\ &= \end{aligned}$$

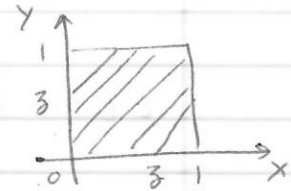
$$\begin{aligned} P(X < Y) &= \int_0^{\infty} \lambda e^{-\lambda x} (1 - (1 - e^{-\mu x})) dx \\ &= \int_0^{\infty} \lambda e^{-\lambda x} e^{-\mu x} dx \\ &= \lambda \left[\frac{e^{-(\lambda+\mu)x}}{-(\lambda+\mu)} \right]_0^{\infty} \\ &= \frac{-\lambda}{\lambda+\mu} [0 - 1] = \frac{\lambda}{\lambda+\mu} \\ &= \end{aligned}$$

Y : lifetime of 100 W bulb $\mu = \frac{1}{100}$
 X : " " 60 W " $\lambda = \frac{1}{200}$

$$P(X < Y) = \frac{\lambda}{\lambda + \mu} = \frac{\frac{1}{200}}{\frac{1}{200} + \frac{1}{100}} = \frac{1}{3}$$

p 279, 10 (4) a) $Z = |X - Y|$

$$\begin{aligned}
 P(Z \leq z) &= P(|X - Y| \leq z) \\
 &= P(-z \leq X - Y \leq z) \\
 &= P(Y \leq X + z \cap Y \geq X - z)
 \end{aligned}$$



$$F_Z(z) = \begin{cases} 1 - (1 - z)^2 & ; z \leq 1 \\ 1 & ; z > 1 \end{cases}$$

$$f_Z(z) = \begin{cases} 2(1 - z) & ; z \leq 1 \\ 0 & ; z > 1 \end{cases}$$

$$\begin{aligned}
 E[|X - Y|] &= \int_0^1 z f_Z(z) dz \\
 &= \int_0^1 2z(1 - z) dz = 2 \left[\frac{z^2}{2} \right]_0^1 - 2 \left[\frac{z^3}{3} \right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}
 \end{aligned}$$

b) $Z = \max(X, Y)$

$$\begin{aligned}
 P(Z \leq z) &= P(\max(X, Y) \leq z) \\
 &= P(X \leq z \cap Y \leq z) \\
 &= P(X \leq z) P(Y \leq z)
 \end{aligned}$$

$$F_Z(z) = \begin{cases} z^2 & ; 0 \leq z \leq 1 \\ 1 & ; z > 1 \end{cases}$$

$$f_Z(z) = \begin{cases} 2z & ; 0 \leq z \leq 1 \\ 0 & ; z > 1 \end{cases}$$

$$E[\max(X, Y)] = \int_0^1 z f_Z(z) dz = \int_0^1 2z^2 dz = 2 \left[\frac{z^3}{3} \right]_0^1 = \frac{2}{3}$$

c) $Z = \min(X, Y)$

$$\begin{aligned}
 P(Z \leq z) &= P(\min(X, Y) \leq z) \\
 &= 1 - P(\min(X, Y) > z) \\
 &= 1 - P(X > z \cap Y > z) \\
 &= 1 - P(X > z) P(Y > z) \\
 &= 1 - (1 - P(X \leq z)) (1 - P(Y \leq z))
 \end{aligned}$$

$$F_Z(z) = \begin{cases} 0 & ; z < 0 \\ 1 - (1-z)^2 & ; 0 \leq z \leq 1 \\ 1 & ; z > 1 \end{cases}$$

$$f_Z(z) = \begin{cases} 0 & ; z < 0 \\ 2(1-z) & ; 0 \leq z \leq 1 \\ 0 & ; z > 1 \end{cases}$$

$$\begin{aligned} E[\min(X, Y)] &= \int_0^1 z f_Z(z) dz = \int_0^1 2z(1-z) dz \\ &= 2 \left[\frac{z^2}{2} \right]_0^1 - 2 \left[\frac{z^3}{3} \right]_0^1 \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

d) $Z = X^2 + Y^2$

$$P(Z \leq z) = P(X^2 + Y^2 \leq z)$$

$$F_Z(z) = \begin{cases} 0 & ; z < 0 \\ \frac{1}{4}\pi z & ; 0 \leq z \leq 1 \\ 1 - A & ; 1 \leq z \leq 2 \\ 1 & ; z \geq 2 \end{cases}$$

$$f_Z(z) = \begin{cases} 0 & ; z < 0 \\ \frac{1}{4}\pi & ; 0 \leq z \leq 1 \\ -\frac{dA}{dz} & ; 1 \leq z \leq 2 \\ 0 & ; z \geq 2 \end{cases}$$

$$A = 1 - \sqrt{z-1} + \frac{3}{2}\sqrt{z-1} - \frac{3\pi}{4} + 3 \tan^{-1} \sqrt{z-1}$$

$$A' = -\frac{1}{2} \frac{1}{\sqrt{z-1}} + \frac{3}{2} \left[\frac{1}{2} (z-1)^{-1/2} \cdot 2 - \sqrt{z-1} \right]$$

$$+ 3 \frac{1}{1+z-1} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{z-1}}$$

$$A' = -\frac{1}{2} \frac{1}{\sqrt{z-1}} + \frac{3}{4\sqrt{z-1}} - \frac{3}{2} \frac{\sqrt{z-1}}{z^2}$$

$$A' = \frac{9}{4\sqrt{z-1}} - \frac{3}{2} \frac{\sqrt{z-1}}{z^2} - \frac{1}{2} \frac{1}{\sqrt{z-1}}$$

$$A = \int_{\sqrt{z-1}}^1 (1 - \sqrt{z-x^2}) dx$$

$$A = 1 - \sqrt{z-1} - \int_{\sqrt{z-1}}^1 \sqrt{z-x^2} dx$$

$$x = \sqrt{z} \sin \theta$$

$$\frac{dx}{d\theta} = \sqrt{z} \cos \theta$$

$$x \rightarrow \sqrt{z-1} \quad \theta \rightarrow \alpha$$

$$x \rightarrow 1 \quad \theta \rightarrow \pi/2 - \alpha$$

$$A = 1 - \sqrt{z-1} - \int_{\alpha}^{\pi/2 - \alpha} \sqrt{z} \cos \theta \sqrt{z} \cos \theta d\theta$$

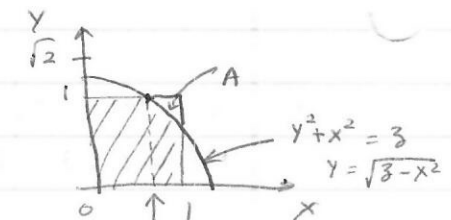
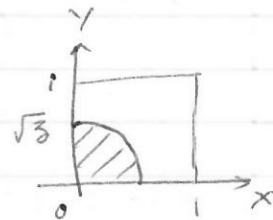
$$A = 1 - \sqrt{z-1} - \frac{z}{2} \int_{\alpha}^{\pi/2 - \alpha} \cos^2 \theta d\theta$$

$$A = 1 - \sqrt{z-1} - \frac{z}{2} \int_{\alpha}^{\pi/2 - \alpha} \cos 2\theta + 1 d\theta$$

$$= 1 - \sqrt{z-1} - \frac{z}{2} \left\{ \frac{\sin 2\theta}{2} \right\}_{\alpha}^{\pi/2 - \alpha} + \left[\theta \right]_{\alpha}^{\pi/2 - \alpha}$$

$$A = 1 - \sqrt{z-1} + \frac{3}{4} \sin 2\alpha - \frac{z}{2} (\pi/2 - 2\alpha)$$

$$A = 1 - \sqrt{z-1} + \frac{3}{4} \frac{\sqrt{z-1}}{z} - \frac{3\pi}{4} + 3 \tan^{-1}(\sqrt{z-1})$$



$$A = \int_{\sqrt{z-1}}^1 (1 - \sqrt{z-x^2}) dx$$

$$x = \sqrt{z} \sin \theta$$

$$\frac{dx}{d\theta} = \sqrt{z} \cos \theta$$

$$x \rightarrow \sqrt{z-1} \quad \theta \rightarrow \alpha = \sin^{-1}(\frac{\sqrt{z-1}}{\sqrt{z}})$$

$$x \rightarrow 1 \quad \theta \rightarrow \pi/2 - \alpha$$

$$A = 1 - \sqrt{z-1} - \int_{\alpha}^{\pi/2 - \alpha} \sqrt{z} \cos \theta \sqrt{z} \cos \theta d\theta$$

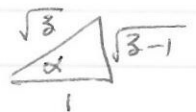
$$A = 1 - \sqrt{z-1} - \frac{z}{2} \int_{\alpha}^{\pi/2 - \alpha} \cos^2 \theta d\theta$$

$$A = 1 - \sqrt{z-1} - \frac{z}{2} \int_{\alpha}^{\pi/2 - \alpha} \cos 2\theta + 1 d\theta$$

$$= 1 - \sqrt{z-1} - \frac{z}{2} \left\{ \frac{\sin 2\theta}{2} \right\}_{\alpha}^{\pi/2 - \alpha} + \left[\theta \right]_{\alpha}^{\pi/2 - \alpha}$$

$$A = 1 - \sqrt{z-1} + \frac{3}{4} \sin 2\alpha - \frac{z}{2} (\pi/2 - 2\alpha)$$

$$A = 1 - \sqrt{z-1} + \frac{3}{4} \frac{\sqrt{z-1}}{z} - \frac{3\pi}{4} + 3 \tan^{-1}(\sqrt{z-1})$$



$$\begin{aligned}
 E(x^2 + y^2) &= \int z f_z(z) dz \\
 &= \int_0^1 \frac{1}{4} \pi z dz + \int_1^2 -z A' dz \\
 &= \frac{1}{4} \pi \left[\frac{z^2}{2} \right]_0^1 - \int_1^2 z A' dz \\
 &= \frac{1}{8} \pi - \int_1^2 z A' dz //
 \end{aligned}$$

c) $z = (x+y)^2$

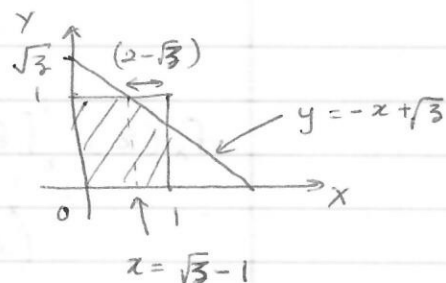
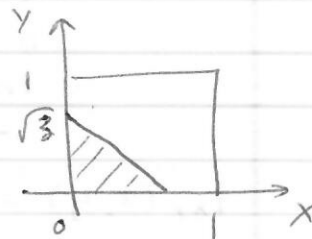
$$\begin{aligned}
 P(Z \leq 3) &= P((x+y)^2 \leq 3) \\
 &= P(x+y \leq \sqrt{3}) \\
 &= P(y \leq -x + \sqrt{3})
 \end{aligned}$$

$$F_z(z) = \begin{cases} 0 & ; z < 0 \\ \frac{1}{2} z & ; 0 \leq z \leq 1 \\ 1 - \frac{1}{2}(2-\sqrt{z})^2 & ; 1 \leq z \leq 4 \\ 1 & ; z \geq 4 \end{cases}$$

$$f_z(z) = \begin{cases} 0 & ; z < 0 \\ \frac{1}{2} & ; 0 \leq z \leq 1 \\ \frac{1}{2}(2-\sqrt{z}) \frac{1}{\sqrt{z}} & ; 1 \leq z \leq 4 \\ 0 & ; z \geq 4 \end{cases}$$

$$E[(x+y)^2] = E[z]$$

$$\begin{aligned}
 &= \int z f_z(z) dz \\
 &= \int_0^1 \frac{1}{2} z dz + \int_1^4 z \left(\frac{1}{\sqrt{z}} - \frac{1}{2} \right) dz \\
 &= \frac{1}{2} \left[\frac{z^2}{2} \right]_0^1 + \frac{2}{3} \left[\frac{z^{3/2}}{3/2} \right]_1^4 - \frac{1}{2} \left[\frac{z^2}{2} \right]_1^4 \\
 &= \frac{1}{4} + \frac{2}{3} (8-1) - \frac{1}{4} (16-1) \\
 &= \frac{1}{4} + \frac{14}{3} - \frac{15}{4} \\
 &= \frac{14}{3} - \frac{14}{4} \\
 &= \frac{7}{6} \\
 &=
 \end{aligned}$$



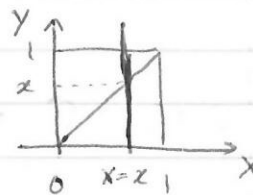
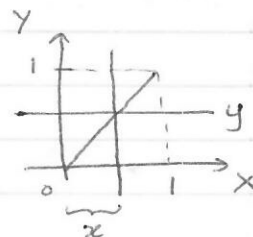
p 282, 22

(5) $Y \sim \text{uni}[0, 1]$ ~~$f_{X,Y}$~~ $X|Y=y \sim \text{uni}[0, y]$

$$\begin{aligned}
 f_X(x) &= \int_0^1 f_{X|Y}(x|y) f_Y(y) dy \\
 &= \int_x^1 \frac{1}{y} dy \\
 &= [\ln y]_x^1 \\
 &= 0 - \ln x \\
 &= \ln(1/x) \\
 &=
 \end{aligned}$$

$$\begin{aligned}
 f_{X,Y}(x, y) &= f_{X|Y}(x|y) f_Y(y) \\
 &= \frac{1}{y} \cdot 1 \\
 &= \frac{1}{y}
 \end{aligned}$$

$$\begin{aligned}
 f_X(x) &= \int_x^1 \frac{1}{y} dy f_{X,Y}(x, y) dy \\
 &= \int_x^1 \frac{1}{y} dy \\
 &= [\ln y]_x^1 \\
 &= \ln(1/x) \\
 &=
 \end{aligned}$$

When $x = x_1 \Rightarrow x \leq y \leq 1$

$$\begin{aligned}
 \textcircled{6} \quad f_X(x) &= \int_0^2 f_{X,Y}(x, y) dy \\
 &= \int_0^2 \frac{1}{2} y e^{-x} dy \\
 &= \frac{1}{2} e^{-x} \left[\frac{y^2}{2} \right]_0^2 \\
 &= \frac{1}{4} e^{-x} (4 - 0) \\
 &= e^{-x} \quad ; x > 0 \\
 &=
 \end{aligned}$$

$$\begin{aligned}
 f_Y(y) &= \int_0^\infty f_{X,Y}(x, y) dx \\
 &= \int_0^\infty \frac{1}{2} y e^{-x} dx \\
 &= \frac{1}{2} y \left[\frac{e^{-x}}{-1} \right]_0^\infty \\
 &= -\frac{1}{2} y (0 - 1) \\
 &= \frac{1}{2} y \quad ; 0 < y < 2 \\
 &=
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{7} \quad P_Y(y) &= \sum_x P_{X,Y}(x, y) \\
 &= \sum_x \frac{1}{15} (x+y) \\
 &= \frac{1}{15} \{ (0+y) + (1+y) + (2+y) \} \\
 &= \frac{1}{15} (1+y)
 \end{aligned}$$

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x, y)}{P_Y(y)} = \frac{\frac{1}{15} (x+y)}{\frac{1}{15} (y+1)} = \frac{1}{3} \frac{(x+y)}{(y+1)} \quad ; x=0, 1, 2 \quad y=1, 2$$

$$p(x=0|Y=2) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$

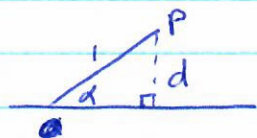
HW 6

① ex. 8. $E[X] = \mu$ $E[X^2] - (E[X])^2 = \sigma^2 \Rightarrow E[X^2] = \sigma^2 + \mu^2$

$$Y = aX^2 + bX + c$$

$$E[Y] = aE[X^2] + bE[X] + c = a(\sigma^2 + \mu^2) + b\mu + c //$$

② ex. 26



$\alpha \sim \text{uniform } [0, \pi]$

$$d = \sin \alpha$$

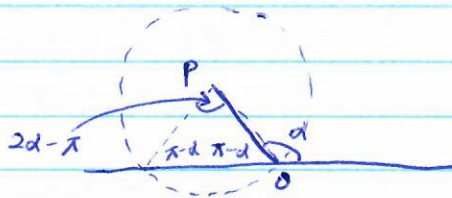
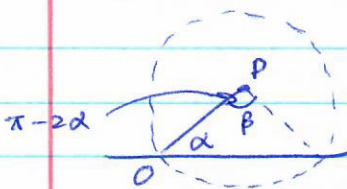
$$E[d] = E[\sin \alpha] = \int_0^\pi \sin \alpha \cdot \frac{1}{\pi} d\alpha = \frac{1}{\pi} [-\cos \alpha]_0^\pi = -\frac{1}{\pi} (-1 - 1) = \frac{2}{\pi}$$

1st angle: $A = \alpha$

2nd angle: Measured from the direction $\vec{PO}$: β

if $\alpha \leq \pi/2$

if $\alpha > \pi/2$



$\beta | A \sim \text{uniform } [0, 2\pi]$

$$P(\text{reach river}) = P(0 \leq \beta \leq \pi - 2\alpha \mid A = \alpha) \text{ if } \alpha \leq \pi/2$$

$$+ P(3\pi - 2\alpha \leq \beta \leq 2\pi \mid A = \alpha) \text{ if } \alpha > \pi/2$$

$$= \int_0^{\pi/2} P(0 \leq \beta \leq \pi - 2\alpha \mid A = \alpha) f_A(\alpha) d\alpha$$

$$+ \int_{\pi/2}^\pi P(3\pi - 2\alpha \leq \beta \leq 2\pi \mid A = \alpha) f_A(\alpha) d\alpha$$

$$= \int_0^{\pi/2} \int_0^{\pi - 2\alpha} \frac{1}{2\pi} d\beta \cdot \frac{1}{\pi} d\alpha + \int_{\pi/2}^\pi \int_{3\pi - 2\alpha}^{2\pi} \frac{1}{2\pi} d\beta \cdot \frac{1}{\pi} d\alpha$$

$$= \int_0^{\pi/2} \frac{1}{2\pi} (\pi - 2\alpha) \frac{1}{\pi} d\alpha + \int_{\pi/2}^\pi \frac{1}{2\pi} (2\pi - (3\pi - 2\alpha)) \frac{1}{\pi} d\alpha$$

$$= \frac{1}{2\pi} \cdot \frac{\pi^2}{2} - \frac{1}{\pi^2} [\alpha^2/2]_0^{\pi/2} + \frac{1}{\pi^2} [\alpha^2/2]_{\pi/2}^\pi - \frac{1}{2\pi} \cdot \frac{\pi^2}{2}$$

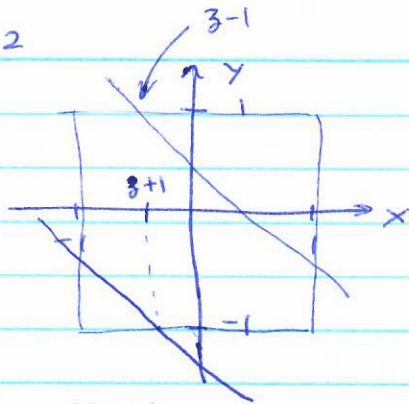
$$= \frac{1}{4} - \frac{1}{2\pi^2} (\frac{\pi^2}{4}) + \frac{1}{2\pi^2} (\pi^2 - \frac{\pi^2}{4}) - \frac{1}{4}$$

$$= -\frac{1}{8} + \frac{3}{8}$$

$$= \frac{1}{4}$$

③ ex. 2

a)



$$f_{xy}(x, y) = 1/4$$

$$z = x + y$$

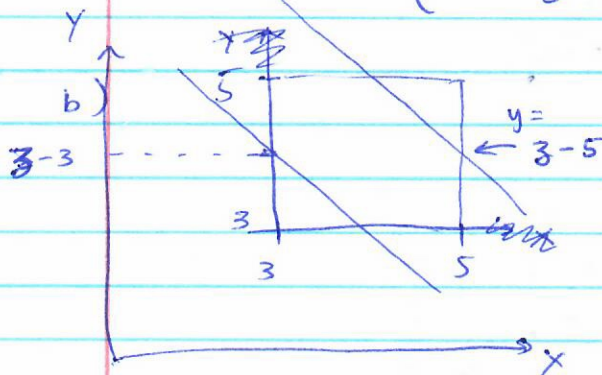
$$P(Z \leq 3) = P(X + Y \leq 3)$$

$$= P(Y \leq -x + 3)$$

$$F_2(z) = \begin{cases} 1/4 \cdot 1/2 \cdot (z+2)^2 & ; -2 \leq z \leq 0 \\ 1 - 1/4 \cdot 1/2 \cdot (2-z)^2 & ; 0 \leq z \leq 2 \\ 1 & ; z \geq 2 \\ 0 & ; \text{o.w.} \end{cases}$$

$$f_2(z) =$$

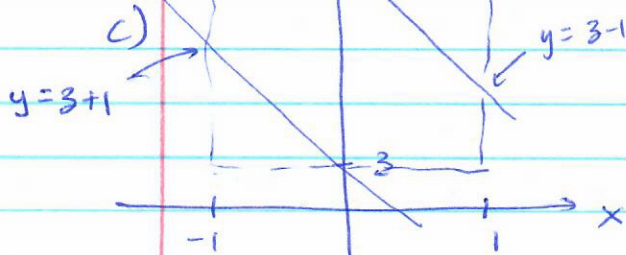
$$f_2(z) = \begin{cases} 1/4 (z+2) & ; -2 \leq z \leq 0 \\ 1/4 (2-z) & ; 0 \leq z \leq 2 \\ 0 & ; \text{o.w.} \end{cases}$$



$$F_2(z) = P(Y \leq -x + 3)$$

$$= \begin{cases} 1/4 \cdot 1/2 (z-6)^2 & ; 6 \leq z \leq 8 \\ 1 - 1/4 \cdot 1/2 (10-z)^2 & ; 8 \leq z \leq 10 \\ 1 & ; z \geq 10 \\ 0 & ; \text{o.w.} \end{cases}$$

$$f_2(z) = \begin{cases} 1/4 (z-6) & ; 6 \leq z \leq 8 \\ 1/4 (10-z) & ; 8 \leq z \leq 10 \\ 0 & ; \text{o.w.} \end{cases}$$



$$F_2(z) = P(Y \leq -x + 3)$$

$$= \begin{cases} 1/4 \cdot 1/2 (z-2)^2 & ; 2 \leq z \leq 4 \\ 1 - 1/4 \cdot 1/2 (6-z)^2 & ; 4 \leq z \leq 6 \\ 1 & ; z \geq 6 \\ 0 & ; \text{o.w.} \end{cases}$$

$$f_2(z) = \begin{cases} 1/4 (z-2) & ; 2 \leq z \leq 4 \\ 1/4 (6-z) & ; 4 \leq z \leq 6 \\ 0 & ; \text{o.w.} \end{cases}$$

HW 7

① Day 1: time of each call: x_i ; $i=1, \dots, 5$

$$D_1 = \frac{x_1 + \dots + x_5}{5} \sim N\left(10, \frac{2^2 \times 5}{25}\right)$$

$$D_1 \sim N(10, 4/5)$$

$$P(\text{Day 1 avg} \leq 11) = P(D_1 \leq 11)$$

$$= P\left(\frac{D_1 - 10}{2/\sqrt{5}} \leq \frac{11 - 10}{2/\sqrt{5}}\right)$$

$$= \Phi(\sqrt{5}/2)$$

Day 2: time of each call: x_i ; $i=1, \dots, 6$

$$D_2 = \frac{x_1 + \dots + x_6}{6} \sim N\left(10, \frac{2^2 \times 6}{36}\right)$$

$$D_2 \sim N(10, 2/3)$$

$$P(\text{Day 2 avg} \leq 11) = P(D_2 \leq 11)$$

$$= P\left(\frac{D_2 - 10}{\sqrt{2/3}} \leq \frac{11 - 10}{\sqrt{2/3}}\right)$$

$$= \Phi(\sqrt{3}/2)$$

$$P(\text{avg of both days} \leq 11) = \Phi(\sqrt{5}/2) \cdot \Phi(\sqrt{3}/2) \quad \text{// (Assuming independent calls)}$$

$$(3) \quad f_{x_1}(x_1) = \lambda_1 e^{-\lambda_1 x_1} \quad f_{x_2}(x_2) = \lambda_2 e^{-\lambda_2 x_2}$$

$$f_{x_1, x_2}(x_1, x_2) = \lambda_1 \lambda_2 e^{-\lambda_1 x_1} e^{-\lambda_2 x_2}$$

$$Z = X_1 + X_2$$

$$F_Z(z) = P(Z \leq z)$$

$$= P(X_1 + X_2 \leq z)$$

$$= P(X_2 \leq -X_1 + z)$$

$$= \int_0^z \int_0^{-x_1+z} f_{x_1, x_2}(x_1, x_2) dx_2 dx_1$$

$$= \int_0^z \int_0^{-x_1+z} \lambda_1 \lambda_2 e^{-\lambda_1 x_1} e^{-\lambda_2 x_2} dx_2 dx_1$$

$$= \lambda_1 \lambda_2 \int_0^z e^{-\lambda_1 x_1} \left[\frac{e^{-\lambda_2 x_2}}{-\lambda_2} \right]_0^{-x_1+z} dx_1$$

$$= -\lambda_1 \int_0^z e^{-\lambda_1 x_1} [e^{-\lambda_2(-x_1+z)} - 1] dx_1$$

$$= -\lambda_1 \int_0^z e^{-(\lambda_1 - \lambda_2)x_1} e^{-\lambda_2 z} - e^{-\lambda_1 x_1} dx_1$$

$$= -\lambda_1 e^{-\lambda_2 z} \left\{ \left[\frac{e^{-(\lambda_1 - \lambda_2)x_1}}{-(\lambda_1 - \lambda_2)} \right]_0^z + \lambda_1 \left[\frac{e^{-\lambda_1 x_1}}{-\lambda_1} \right]_0^z \right\}$$

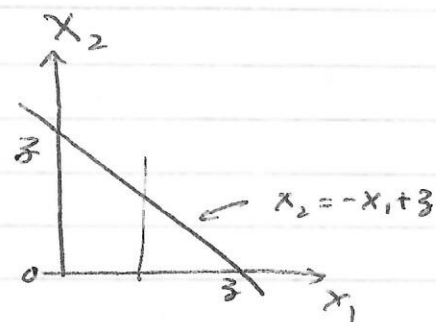
$$= \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 z} [e^{-(\lambda_1 - \lambda_2)z} - 1] - (e^{-\lambda_1 z} - 1)$$

$$= \frac{\lambda_1}{\lambda_1 - \lambda_2} [e^{-\lambda_1 z} - e^{-\lambda_2 z}] - e^{-\lambda_1 z} + 1$$

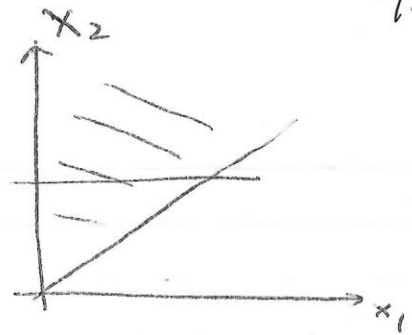
$$f_z(z) = \frac{\lambda_1}{\lambda_1 - \lambda_2} [-\lambda_1 e^{-\lambda_1 z} + \lambda_2 e^{-\lambda_2 z}] + \lambda_1 e^{-\lambda_1 z}$$

$$= \frac{1}{\lambda_1 - \lambda_2} [-\lambda_1^2 e^{-\lambda_1 z} + \lambda_1 \lambda_2 e^{-\lambda_2 z} + \lambda_1 (\lambda_1 - \lambda_2) e^{-\lambda_1 z}]$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} [e^{-\lambda_2 z} - e^{-\lambda_1 z}] \quad , \quad z \geq 0$$



$$Y = \frac{X_1}{X_1 + X_2}$$



$$F_Y(y) = P(Y \leq y)$$

$$= P\left(\frac{X_1}{X_1 + X_2} \leq y\right)$$

$$= P(X_1 \leq y(X_1 + X_2))$$

$$= P(X_2 \geq \frac{X_1(1-y)}{y})$$

$$= \begin{cases} \int_0^{\infty} \int_0^{\frac{y}{1-y}x_2} f_{X_1, X_2}(x_1, x_2) dx_1 dx_2 & ; 0 \leq \frac{1-y}{y} \Rightarrow y \leq 1 \\ 1 & ; y \geq 1 \end{cases}$$

$$\begin{aligned} \int_0^{\infty} \int_0^{\frac{y}{1-y}x_2} f_{X_1, X_2}(x_1, x_2) dx_1 dx_2 &= \int_0^{\infty} \int_0^{\frac{y}{1-y}x_2} \lambda_1 \lambda_2 e^{-\lambda_1 x_1} e^{-\lambda_2 x_2} dx_1 dx_2 \\ &= \lambda_1 \lambda_2 \int_0^{\infty} e^{-\lambda_2 x_2 \left(\frac{y}{1-y}\right)} e^{-\lambda_1 x_1} dx_1 dx_2 \\ &= \lambda_1 \lambda_2 \int_0^{\infty} e^{-\lambda_2 x_2} \left[\frac{e^{-\lambda_1 x_1}}{-\lambda_1} \right]_0^{\frac{y}{1-y}x_2} dx_2 \\ &= -\lambda_2 \int_0^{\infty} e^{-\lambda_2 x_2} \left[e^{-\lambda_1 x_2 \left(\frac{y}{1-y}\right)} - 1 \right] dx_2 \\ &= -\lambda_2 \int_0^{\infty} e^{-x_2(\lambda_2 + \lambda_1 \left(\frac{y}{1-y}\right))} - e^{-\lambda_2 x_2} dx_2 \\ &= -\lambda_2 \left[\frac{e^{-x_2(\lambda_2 + \lambda_1 \left(\frac{y}{1-y}\right))}}{-(\lambda_2 + \lambda_1 \left(\frac{y}{1-y}\right))} \right]_0^{\infty} + \lambda_2 \left[\frac{e^{-\lambda_2 x_2}}{-\lambda_2} \right]_0^{\infty} \\ &= \frac{\lambda_2}{\lambda_2 + \lambda_1 \left(\frac{y}{1-y}\right)} [0 - 1] - 1 [0 - 1] \end{aligned}$$

$$F_Y(y) = 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1 \left(\frac{y}{1-y}\right)} ; y \leq 1$$

$$f_Y(y) = - \left[\frac{-\lambda_1 [1-y + y/(1-y)^2] \lambda_2}{(\lambda_2 + \lambda_1 \left(\frac{y}{1-y}\right))^2} \right]$$

$$= \frac{\lambda_1 \lambda_2}{(1-y)^2 (\lambda_2 + \lambda_1 \left(\frac{y}{1-y}\right))^2}$$

$$= \frac{\lambda_1 \lambda_2}{(\lambda_2(1-y) + \lambda_1 y)^2}$$

$$f_Y(y) = \frac{\lambda_1 \lambda_2}{[\lambda_2 + y(\lambda_1 - \lambda_2)]^2} ; y \leq 1$$

④ $X_i \sim \text{poi}(1)$
 $\sum_{i=1}^{20} X_i \sim \text{poi}(20)$

a) $P\left(\sum_{i=1}^{20} X_i > 15\right) \leq \frac{E\left[\sum_{i=1}^{20} X_i\right]}{15} = \frac{\sum_{i=1}^{20} E[X_i]}{15} = \frac{20}{15} = \frac{4}{3}$

b) $P\left(\sum_{i=1}^{20} X_i > 15\right) \approx P\left(\frac{\sum_{i=1}^{20} X_i - 20\mu}{\sqrt{20\text{Var}(X_i)}} > \frac{15 - 20\mu}{\sqrt{20\text{Var}(X_i)}}\right)$
 $= P\left(\frac{\sum_{i=1}^{20} X_i - 20}{\sqrt{20}} > \frac{15 - 20}{\sqrt{20}}\right)$
 $= 1 - \Phi(-\sqrt{5}/2)$
 $= 1 - (1 - \Phi(\sqrt{5}/2))$
 $= \Phi(\sqrt{5}/2)$

c) $P\left(\sum_{i=1}^{20} X_i > 15\right) = 1 - \sum_{j=0}^{15} P\left(\sum_{i=1}^{20} X_i = j\right)$
 $= 1 - \sum_{j=0}^{15} \left(\frac{e^{-20} (20)^j}{j!}\right)$
 $= 1 - e^{-20} \sum_{j=0}^{15} \frac{20^j}{j!}$

⑤ true value = ℓ
 i^{th} measurement: L_i
 $L_i - \ell \sim \text{uniform}[-1, 1]$
 $L = \frac{\sum_{i=1}^{50} L_i}{50}$

$P(|L - \ell| \leq 0.25) = P(-0.25 \leq L - \ell \leq 0.25)$
 $= P(-0.25 \leq \frac{\sum_{i=1}^{50} (L_i - \ell)}{50} \leq 0.25)$

Problem 5. Let X_i be the outcome of the i^{th} measurement.

$$X_i \sim \text{Unif}[-1, 1], E X_i = 0, \text{Var}(X_i) = \frac{2}{12} = \frac{1}{6}$$

We need to find

$$P\left(-0.25 < \frac{\sum_{i=1}^n X_i}{n} < 0.25\right) = P\left(-\frac{0.25\sqrt{n}}{\sigma} < \frac{\sum X_i}{\sqrt{n}\sigma} < \frac{0.25\sqrt{n}}{\sigma}\right)$$

where $n=50$; $\sigma = \sqrt{1/6}$.

The RV $Z = \frac{0.25\sqrt{n}}{\sigma}$ is approx. standard normal, so

$$P(|Z| < 0.25\sqrt{\frac{50}{6}}) \approx 1 - 2\Phi(-0.295) \approx 0.232 //$$

Problem 2. As in Problem 5, we have $E X_i = \mu = 0.18$; $\text{Var } X_i = \sigma^2 = 0.06$
 $i=1, \dots, n=40$

Since X_i for different i are independent,

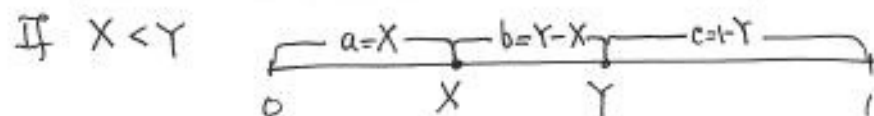
$Z \triangleq \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma}$ is approx standard normal, so

$$P\left(0.16 < \frac{\sum_{i=1}^n X_i}{n} < 0.19\right) = P\left(0.16n < \sum_{i=1}^n X_i < 0.19n\right)$$

$$= P\left(\frac{0.16n - n\mu}{\sigma\sqrt{n}} < Z < \frac{0.19n - n\mu}{\sigma\sqrt{n}}\right) = P\left(\frac{-0.02 \cdot 40}{0.06\sqrt{40}} < Z < \frac{0.01 \cdot 40}{0.06\sqrt{40}}\right)$$

$$\approx \Phi(1.05) - \Phi(-2.11) \approx 0.836 //$$

Problem 1. If a (or b , or c) is greater than $\frac{1}{2}$, then $a > b+c$, so a, b, c cannot form a triangle. Thus, $0 < a < \frac{1}{2}$; $0 < b < \frac{1}{2}$; $0 < c < \frac{1}{2}$. Next, let X and Y be uniform random points on $[0, 1]$.



then we can form a triangle if

$$c < \frac{1}{2}, \text{ i.e., } Y > \frac{1}{2}$$

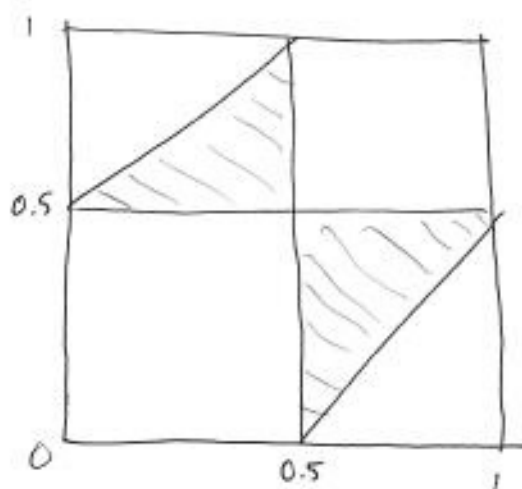
$$b < \frac{1}{2}, \text{ i.e., } Y < X + \frac{1}{2}$$

$$a < \frac{1}{2}, \text{ i.e., } X < \frac{1}{2}$$

If $X > Y$, we similarly obtain the conditions

$$X > \frac{1}{2}, X < Y + \frac{1}{2}, Y < \frac{1}{2}$$

Since the pair (X, Y) is selected uniformly in the $(0, 1)$ square, the required probability is calculated as the area of the two regions indicated above, and shown in the figure below.



The area of these two triangles is $\frac{1}{4}$.

Answer: $\frac{1}{4}$

HW 8

$$(2) p_X(k) = 2\left(\frac{1}{3}\right)^k, \quad k=1, 2, \dots$$

$$M.G.F. = E[e^{tX}]$$

$$= \sum_{k=1}^{\infty} e^{tX} \cdot 2\left(\frac{1}{3}\right)^k$$

$$= 2 \sum_{k=1}^{\infty} \left(\frac{1}{3}e^t\right)^k$$

$$= 2 \left\{ \left(\frac{1}{3}e^t\right)^1 + \left(\frac{1}{3}e^t\right)^2 + \dots \right\}$$

$$= 2 \left(\frac{1}{3}e^t\right)$$

$$1 - \frac{1}{3}e^t$$

$$g_X(s) = \frac{2}{3e^s - 1}$$

$$E[X] = g'_X(s) \Big|_{s=0} = \frac{+3e^{-s} \cdot 2}{(3e^{-s} - 1)^2} \Big|_{s=0} = 6$$

$$E[X] = 6$$

$$(3) g_X(s) = \frac{1}{3}e^s + \frac{4}{15}e^{3s} + \frac{2}{15}e^{4s} + \frac{4}{15}e^{5s}$$

$$f_X(x) = \int_{-\infty}^{\infty} g_X(s) e^{-sx} ds$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{3}e^s + \frac{4}{15}e^{3s} + \frac{2}{15}e^{4s} + \frac{4}{15}e^{5s}\right) e^{-sx} ds$$

$$= \frac{1}{3} \int_{-\infty}^{\infty} e^{(1-x)s} ds + \frac{4}{15} \int_{-\infty}^{\infty} e^{(3-x)s} ds + \frac{2}{15} \int_{-\infty}^{\infty} e^{(4-x)s} ds + \frac{4}{15} \int_{-\infty}^{\infty} e^{(5-x)s} ds$$

$$f_X(x) = \frac{1}{3} \delta(x-1) + \frac{4}{15} \delta(x-3) + \frac{2}{15} \delta(x-4) + \frac{4}{15} \delta(x-5)$$

$$(4) g_X(s) = \frac{1}{(1-s)^2}$$

$$g'_X(s) = -2(1-s)^{-3}(-1) = \frac{2}{(1-s)^3}$$

$$g''_X(s) = -6(1-s)^{-4}(-1) = \frac{3!}{(1-s)^4}$$

$$g_X^{(n)}(s) = \frac{(n+1)!}{(1-s)^{n+2}}$$

$$E[X^n] = g_X^{(n)}(s) \Big|_{s=0} = (n+1)! //$$

$$\begin{aligned}
 \textcircled{5} \quad E[XY] &= \int_0^{\infty} \int_0^{\infty} xy f_{xy}(x, y) dx dy \\
 &= \int_0^{\infty} \int_0^{\infty} \frac{1}{2} x^2 y e^{-xy-x} dx dy \\
 &= \int_0^{\infty} \frac{1}{2} x^2 e^{-x} \int_0^{\infty} y e^{-xy} dy dx \\
 &= \int_0^{\infty} \frac{1}{2} x^2 e^{-x} \left\{ \frac{1}{x^2} \right\} dx \\
 &= \frac{1}{2} \int_0^{\infty} x^2 e^{-x} dx \\
 &= \frac{1}{2} \cdot 2! \\
 &= 1 \quad \text{positively correlated.}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{6} \quad \text{Cov}(X, Y) &= E[(X - \bar{X})(Y - \bar{Y})] \\
 f_{xy}(x, y) &= \sin x \cdot \sin y \\
 f_x(x) &= \int_0^{\pi/2} \sin x \cdot \sin y \cdot dy = \sin x [-\cos y]_0^{\pi/2} \\
 &= -\sin x (0 - 1) \\
 &= \sin x
 \end{aligned}$$

$$\begin{aligned}
 f_y(y) &= \int_0^{\pi/2} \sin x \cdot \sin y \cdot dx = \sin y [-\cos x]_0^{\pi/2} = -\sin y (0 - 1) = \sin y \\
 \bar{X} &= E[X] = \int_0^{\pi/2} x \sin x \cdot dx = 1
 \end{aligned}$$

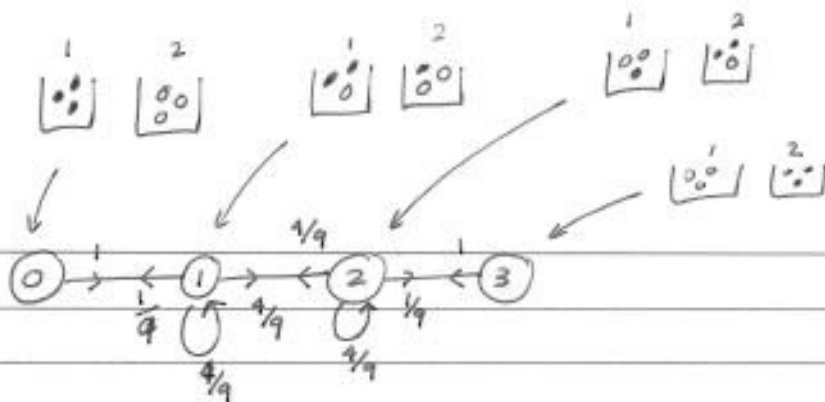
Similarly $\bar{Y} = 1$

$$\text{Var}(X) = E[X^2] - \bar{X}^2 = \pi - 2 - 1 = \pi - 3$$

$$\text{Var}(Y) = \pi - 3$$

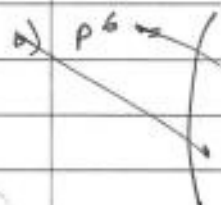
$$\begin{aligned}
 \text{Cov}(X, Y) &= E[XY] - \bar{X} \bar{Y} \\
 &= \int_0^{\pi/2} \int_0^{\pi/2} xy \sin x \sin y \cdot dx dy - \bar{X} \bar{Y} \\
 &= \underbrace{\int_0^{\pi/2} x \sin x \cdot dx}_{\bar{X}} \underbrace{\int_0^{\pi/2} y \sin y \cdot dy}_{\bar{Y}} - \bar{X} \bar{Y} \\
 &= 0
 \end{aligned}$$

$$\rho = 0$$



$$a) P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$b) P(X_6=3 | X_0=3) = P(3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0) + P(3 \rightarrow 1 \rightarrow 2 \rightarrow 2 \rightarrow 2 \rightarrow 1 \rightarrow 0) + P(3 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 0) + P(3 \rightarrow 1 \rightarrow 2 \rightarrow 1)$$



$$b) P^6(3, 3) =$$

$$b) P(X_6=3 | X_0=3) = P^6(3, 3) = 0.051$$

$$8) P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 5/6 & 0 & 1/6 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 17/36 & 2/9 & 11/36 \\ 5/9 & 2/9 & 2/9 \\ 5/12 & 5/18 & 11/36 \end{pmatrix}$$

$$E[X_3] = \sum_{x_3=0}^2 x_3 P(X_3=x_3)$$

$$= \sum_{x_3=0}^2 x_3 \sum_{x_0=0}^2 P(X_3=x_3 | X_0=x_0) P(X_0=x_0)$$

$$= \{P(X_3=1 | X_0=0)P(X_0=0) + P(X_3=1 | X_0=1)P(X_0=1) + P(X_3=1 | X_0=2)P(X_0=2)\}$$

$$+ 2\{P(X_3=2 | X_0=0)P(X_0=0) + P(X_3=2 | X_0=1)P(X_0=1) + P(X_3=2 | X_0=2)P(X_0=2)\}$$

$$= \{2/9 \cdot 1/4 + 2/9 \cdot 1/4 + 5/18 \cdot 1/2\} + 2\{11/36 \cdot 1/4 + 2/9 \cdot 1/4 + 11/36 \cdot 1/2\}$$

$$= 9/36 + 2(37/72)$$

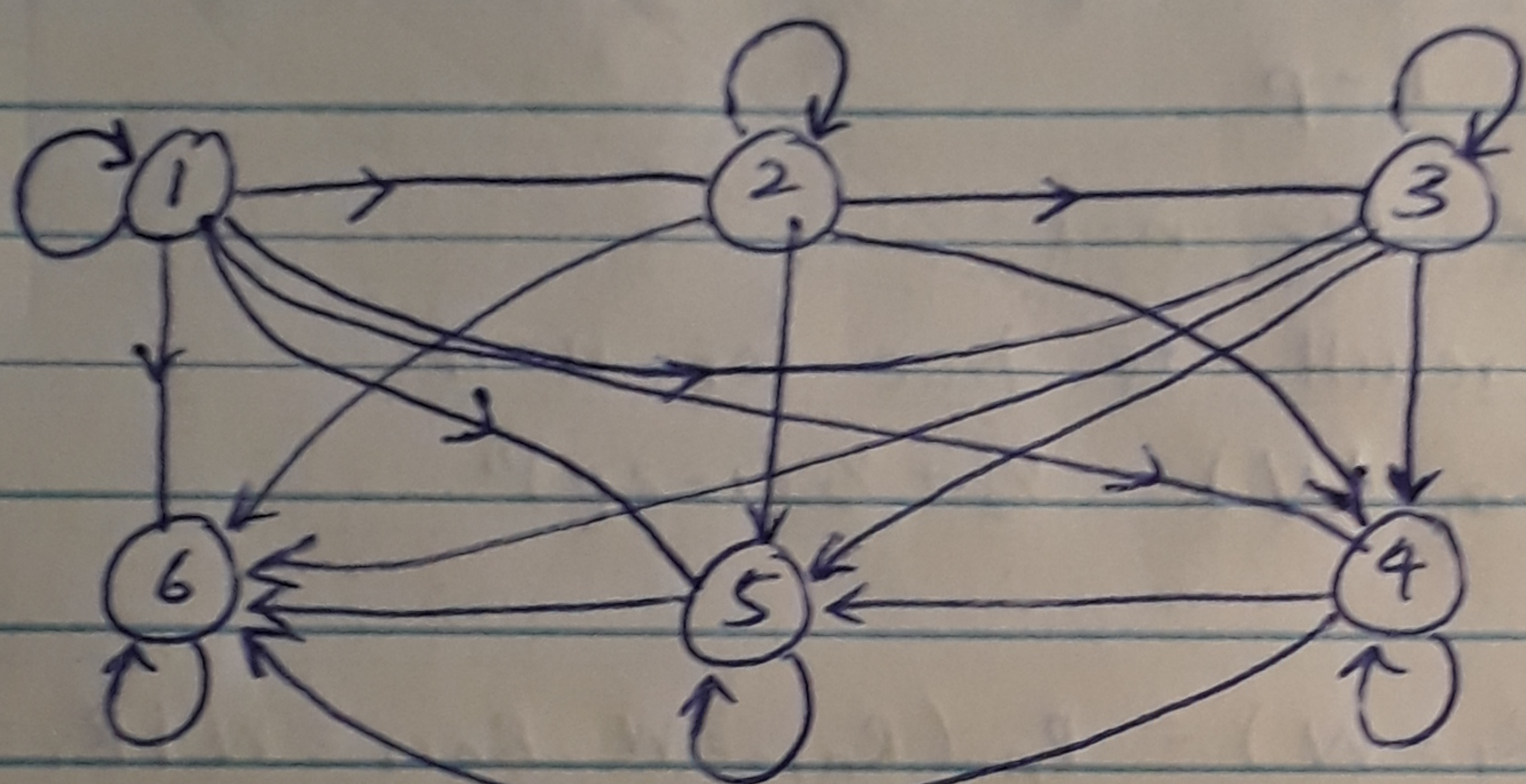
$$= 23/18$$

$$\frac{22}{36} + \frac{1}{4}$$

$$\frac{22}{72} + \frac{1}{18} = \frac{23}{72}$$

HW 9

- ① $X_1 = R_1$: R_i - outcome of i^{th} roll
 $X_2 = \max(X_1, R_2)$
 $X_3 = \max(X_2, R_3)$
 $\vdots$
 $X_n = \max(X_{n-1}, R_n)$



Yes. Current state only depends on the previous state

$$P(X_{n+1} = i | X_n = i) = P(R_{n+1} \leq i | X_n = i) = \frac{i}{6}$$

$$P(X_{n+1} = R_{n+1} | X_n = i) = \frac{1}{6}$$

$$P = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

recurrent ⑥

transient ① ② ③ ④ ⑤

closed class ⑥, {⑤, ⑥},
 {④, ⑤, ⑥},
 {③, ④, ⑤, ⑥},
 {②, ③, ④, ⑤, ⑥}

③

round ~~n~~

round ~~n+1~~

$$P_A(W_{n+1})$$

$$\left. \begin{aligned} P_A(R_{n+1} = W \mid R_n = L) &= p \\ P_A(R_{n+1} = L \mid R_n = W) &= p \end{aligned} \right\} \textcircled{1}$$

for $n=1$ $\rightarrow$ in $\textcircled{1}$: $P(R_2 = W \mid R_1 = W) = 1-p$

$$\frac{1}{2} + \frac{1}{2}(1-2p)^1 = 1-p$$

$\therefore$ result is true for $n=1$

Assume that the result is true for $n=n$

$$P_A(R_{n+1} = W \mid R_n = W) = \frac{1}{2} + \frac{1}{2}(1-2p)^n$$

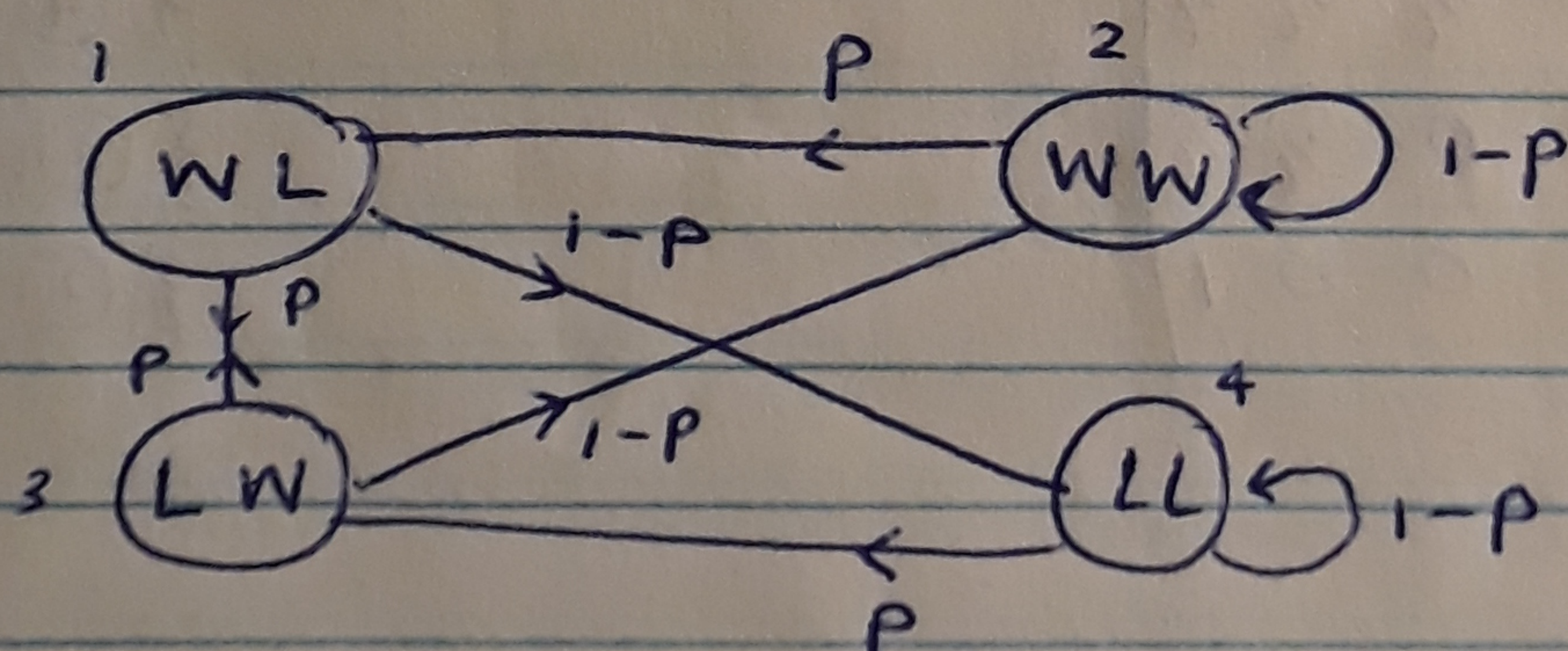
for $n=n+1$

$$\begin{aligned} P_A(R_{n+2} = W \mid R_1 = W) &= P_A(R_{n+2} = W, R_{n+1} = W \mid R_1 = W) \\ &\quad + P_A(R_{n+2} = W, R_{n+1} = L \mid R_1 = W) \\ &= P_A(R_{n+2} = W \mid R_{n+1} = W, R_1 = W) P(R_{n+1} = W \mid R_1 = W) \\ &\quad + P_A(R_{n+2} = W \mid R_{n+1} = L, R_1 = W) P(R_{n+1} = L \mid R_1 = W) \\ &= (1-p) \left[\frac{1}{2} + \frac{1}{2}(1-2p)^n \right] + p \left[1 - \frac{1}{2} - \frac{1}{2}(1-2p)^n \right] \\ &= \left(\frac{1}{2} + \frac{1}{2}(1-2p)^n \right) (1-p-p) + p \\ &= \frac{1}{2}(1-2p) + \frac{1}{2}(1-2p)^{n+1} + p \\ &= \frac{1}{2} + \frac{1}{2}(1-2p)^{n+1} \end{aligned}$$

$\therefore$ result is true for $n=n+1$

$\Rightarrow$ Proven by Mathematical Induction.

b)



t_i : expected time to reach $\textcircled{2}$ from state i

$$t_1 = p(t_3+1) + (1-p)(t_4+1) = t_3 p + (1-p)t_4 + 1 \quad \text{---} \textcircled{1}$$

$$t_2 = (1-p)(t_1+1) + p(t_1+1) = (1-p)t_1 + p t_1 + 1 \quad \text{---} \textcircled{2}$$

$$t_3 = (1-p)(t_2+1) + p(t_1+1) = (1-p)t_2 + p t_1 + 1 \quad \text{---} \textcircled{3}$$

$$t_4 = p(t_3+1) + (1-p)(t_4+1) = p t_3 + (1-p)t_4 + 1 \quad \text{---} \textcircled{4}$$

$$pt_4 = pt_3 + 1$$

$$\begin{aligned} t_1 &= t_3 p + (1-p)(t_3 + \frac{1}{p}) + 1 \\ &= t_3 + \frac{1}{p} - 1 + 1 \\ &= t_3 + \frac{1}{p} \end{aligned}$$

~~$$pt_2 = pt_1 + 1$$~~

$$t_3 = (1-p)t_2 + pt_3 + 2$$

$$\begin{aligned} (1-p)t_3 &= (1-p)t_2 + 2 \quad \text{--- (1)} \\ &= (1-p)(pt_1 + 1) + 2 \\ &= p(1-p)t_1 + 1 - p + 2 \\ &= p(1-p)(t_3 + \frac{1}{p}) - p + 3 \\ &= p(1-p)t_3 + 1 - 2p + 3 \end{aligned}$$

$$(1-p)t_3(1-p) = 4 - 2p$$

$$t_3 = \frac{2(2-p)}{(1-p)^2}$$

$$\text{from (1)} \quad t_2 = \frac{(1-p)t_3 - 2}{1-p} = \frac{\frac{2(2-p)}{(1-p)} - 2}{(1-p)}$$

$$t_2 = \frac{2(2-p) - 2(1-p)}{(1-p)^2} = \frac{2}{(1-p)^2}$$

$$\begin{aligned}
 \textcircled{4} \quad \text{a) } P(N(\tfrac{1}{5}) = 2 \mid N(1) = 2) &= \frac{P(N(\tfrac{1}{5}) = 2, N(1) = 2)}{P(N(1) = 2)} \\
 &= \frac{P(N(\tfrac{1}{5}) = 2, N(1) - N(\tfrac{1}{5}) = 0)}{P(N(1) = 2)} \\
 &= \frac{P(N(\tfrac{1}{5}) = 2) P(N(1) - N(\tfrac{1}{5}) = 0)}{P(N(1) = 2)} \\
 &= \frac{e^{-\frac{1}{5}\lambda} (\frac{1}{5}\lambda)^2}{2!} \frac{e^{-\frac{4}{5}\lambda} (\frac{4}{5}\lambda)^0}{0!} / \frac{e^{-\lambda} \lambda^2}{2!} \\
 &= \frac{1}{5} \cdot \frac{1}{9} = \frac{1}{45}
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } P(N(\tfrac{1}{5}) \geq 1 \mid N(1) = 2) &= 1 - P(N(\tfrac{1}{5}) = 0 \mid N(1) = 2) \\
 &= 1 - \frac{P(N(\tfrac{1}{5}) = 0, N(1) = 2)}{P(N(1) = 2)} \\
 &= 1 - \frac{P(N(\tfrac{1}{5}) = 0, N(1) - N(\tfrac{1}{5}) = 2)}{P(N(1) = 2)} \\
 &= 1 - \frac{P(N(\tfrac{1}{5}) = 0) P(N(1) - N(\tfrac{1}{5}) = 2)}{P(N(1) = 2)} \\
 &= 1 - \frac{e^{-\frac{1}{5}\lambda} (\frac{1}{5}\lambda)^0}{0!} \times \frac{e^{-\frac{4}{5}\lambda} (\frac{4}{5}\lambda)^2}{2!} / \frac{e^{-\lambda} \lambda^2}{2!} \\
 &= 1 - \frac{1}{5} \cdot \frac{16}{9} = \frac{4}{9} \\
 &= \frac{4}{9}
 \end{aligned}$$

$$\textcircled{5} \quad P(N(1) \geq 1 \mid N(4) = 6) = \frac{P(N(1) \geq 1, N(4) = 6)}{P(N(4) = 6)}$$

$$= P(N(1) \geq 1)$$

$$P(N(4) = 6) = \frac{e^{-4\lambda} (4\lambda)^6}{6!}$$

$$P(N(1) \geq 1, N(4) = 6) = \sum_{t=1}^6 P(N(1) = t, N(4) - N(1) = 6 - t)$$

$$= \sum_{t=1}^6 P(N(1) = t) P(N(4) - N(1) = 6 - t)$$

$$= \sum_{t=1}^6 \frac{e^{-\lambda} \lambda^t}{t!} \frac{e^{-3\lambda} (3\lambda)^{6-t}}{(6-t)!}$$

$$= e^{-4\lambda} \lambda^6 \sum_{t=1}^6 \frac{3^{6-t}}{t! (6-t)!}$$

$$= \lambda^6 e^{-4\lambda} \left\{ \frac{1}{6!} + \frac{3}{5!} + \frac{3^2}{4!2!} + \frac{3^3}{3!3!} + \frac{3^4}{2!4!} + \frac{3^5}{5!} \right\}$$

$$P(N(1) \geq 1 \mid N(4) = 6) = \frac{6!}{4^6} \left\{ \frac{1}{6!} + \frac{3}{5!} + \frac{3^2}{4!2!} + \frac{3^3}{3!3!} + \frac{3^4}{2!4!} + \frac{3^5}{5!} \right\}$$

$$= \frac{1}{4^6} \left\{ 1 + 18 + \frac{3^2 \cdot 6 \cdot 5}{2} + \frac{3^3 \cdot 6 \times 5 \times 4}{3 \times 2} + \frac{3^4 \cdot 6 \cdot 5}{2} + 3^5 \cdot 6 \right\}$$

$$= \frac{1}{4^6} \{ 19 + 3^3 \cdot 5 + 3^3 \cdot 20 + 3^5 \cdot 5 + 3^5 \cdot 6 \}$$

$$= \underline{\underline{0.82}}$$