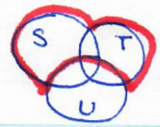
ENEE 324 HW1 Solutions

1. (i) $S \cap T \cap U^c$



(ii) $((S\cap T)\cup(T\cap U)\cup(S\cap U))^c$
$= (S\cap T)^c \cap (T\cap U)^c \cap (S\cap U)^c$

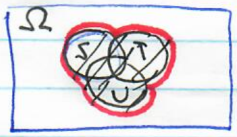
complement of highlighted set

(iii) $(S\cap T)\cup(T\cap U)\cup(S\cap U)$
this subset is highlighted in the Venn diagram in part (ii)

(iv) $((S\cap T)\cup(T\cap U)\cup(S\cap U))\setminus(S\cap T\cap U)$

note that the triple intersection is not included

(v) $(S\cup T\cup U)^c = S^c \cap T^c \cap U^c$



complement of the highlighted set

Sample space $\Omega = 216$ possible outcomes

2. $P(R_1+R_2+R_3 \text{ even}) = \frac{1}{2}$

Indeed, for every outcome of the first two rolls, (R_1,R_2) there are 3 out of 6 possible R_3 that lead to the even sum $R_1+R_2+R_3$
Thus, the total # of even outcomes $= 36\cdot 3 = 108$.
Since every outcome has probability $\frac{1}{|\Omega|} = \frac{1}{216}$, the answer is $\frac{108}{216} = \frac{1}{2}$

$P(\cancel{R_1}+R_2+R_3 \text{ even}) = \frac{1}{2}$
Here we take a subset of Ω formed of all 36 pairs (R_2,R_3), of them 18 sum to an even number.

$P(R_1+R_2+R_3=5) = \frac{6}{216} = \frac{1}{36}$
The event in question is as follows

R_1	1	1	3	1	2	2
R_2	1	3	1	2	1	2
R_3	3	1	1	2	2	1

3. The largest possible value of π is when $B \subset A$, then $\pi = P(B) = 0.6$. The smallest possible value is found as follows:
By normalization,
$$P(A\setminus B) + P(B\setminus A) + P(A\cap B) \leq 1$$
$$P(A)+P(B) = P(A\setminus B) + P(B\setminus A) + P(A\cap B) + P(A\cap B) \leq 1 + P(A\cap B)$$
$$\Rightarrow P(A\cap B) \geq P(A)+P(B)-1 = 0.35.$$

Answer: (a) and (d) are always correct

(4)

$\Omega =$

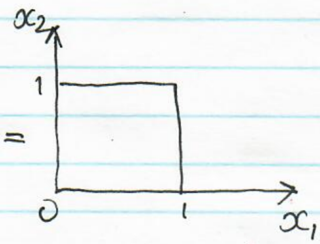
11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

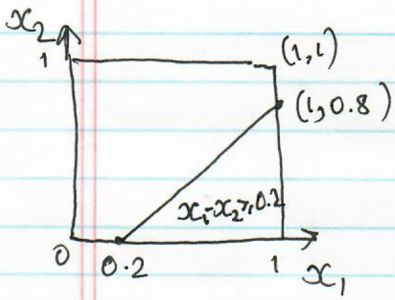
$R_1 \geq R_2 + 2$

There are 15 possibilities for $R_1 > R_2$, so

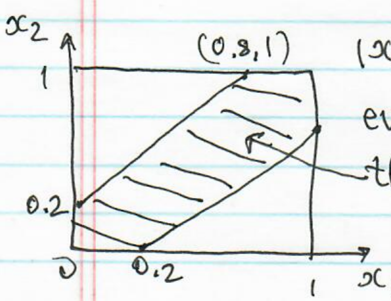
$$P(R_1 > R_2) = \frac{15}{36} = \frac{5}{12}$$

Similarly $P(R_1 \geq R_2 + 2) = \frac{10}{36} = \frac{5}{18}$

(5) The sample space $\Omega =$  is the unit square



$$P(x_1 - x_2 \geq 0.2) = \frac{\text{area of } \triangle}{\text{area of } \Omega} = 0.32$$



$|x_1 - x_2| \geq 0.2$

everything but this region

$$P(|x_1 - x_2| \geq 0.2) = 0.64$$

(6) $P(KK) = \frac{4}{52} \cdot \frac{3}{51} = \cancel{\frac{3}{653}}$ 3/663=1/221

$P(\text{diff. values}) = 1 - P(\text{identical values}) = 1 - P(C_2 = C_1 \mid C_1) =$

$$= 1 - \frac{3}{51} = \frac{48}{51} = \frac{16}{17}$$

ENEE324 Fall 2016. Problem set 2, Solutions.

① (a) Let $A = \{F_1 = F_2 = \dots = F_n = T\}$. The event in question is A^c, so

$$P(A^c) = 1 - P(A) = 1 - 2^{-n}$$

(b) There are $\binom{n}{k}$ different ways to choose k locations of H, so the required probability is $\binom{n}{k} 2^{-n}$.

(c) Suppose that $F_i F_{i+1} F_{i+2}$ is the first place where the outcomes are HHH. If $i \geq 2$, then $F_{i-1} = T$, but then $(F_{i-1} F_i F_{i+1}) = THH$, so THH appears before HHH. Thus, the only way for HHH to appear earlier is $i = 1$, and the probability of that is $\frac{1}{8}$.

(d) ~~Let~~ $\cancel{A_i = \{F_i = H\}, A = \bigcap_{i=1}^{\infty} A_i}$. ~~Then~~

$$\cancel{P(A) = \lim_{n\to\infty} P\Big(\bigcap_{i=1}^{n} A_i\Big) =}$$

Let $A_i = \{F_1 = \dots = F_i = H$, anything after that$\}$, i.e., the first i tosses are H.

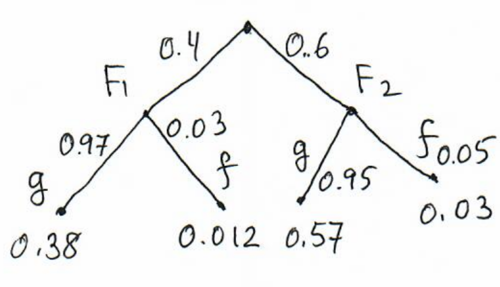
The event in question is $A = \bigcap_{i=1}^{\infty} A_i$

$$P(A) = \lim_{i\to\infty} P(A_i) = \lim_{i\to\infty} 2^{-i} = 0.$$

The sample space is $\Omega = \{$all infinite sequences of H and T$\}$.

The cardinality of Ω is the same as of the set of infinite sequences of 0's and 1's, i.e., the same as of the set $[0,1]$, i.e. same as $\mathbb{R}$.

②.



From the tree diagram

$p = \text{Prob (faulty switch)} = 0.012 + 0.03 = 0.042$

$1 - p = 0.958$

Among 5 switches, there are 5 possibilities for one to be faulty

Answer: $5p(1-p)^4 = 5 \cdot 0.015 \cdot 0.958^4 \cong 0.063$

(3.) Compute $p = P(2^{nd}$ card is smaller than $1^{st})$

Using the total probability theorem, let $A_i = \{1^{st} \text{ card} = i\}$, $i = 1, 2, \dots, n$, and write the required probability as

$$p = \sum_{i=2}^{n} P(2^{nd} < 1^{st} \mid A_i) P(A_i) = \sum_{i=2}^{n} \frac{i-1}{n-1} \cdot \frac{1}{n} = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} i = \frac{1}{n(n-1)} \frac{n(n-1)}{2} = \frac{1}{2}$$

The probability of the event $\{1^{st} < 2^{nd}\} = 1 - p = \frac{1}{2}$.

(4) Let U_1 be the event that the first urn is selected, and let A be the event that the 3 marbles drawn are blue. Then

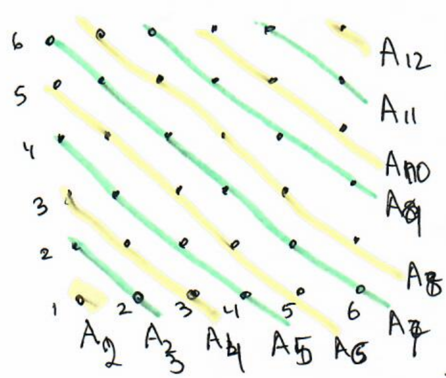
$$P(U_1 \mid A) = \frac{P(U_1 A)}{P(A)} = \frac{P(A \mid U_1) P(U_1)}{P(A \mid U_1) P(U_1) + P(A \mid U_2) P(U_2)} = \frac{4/33 \cdot 1/2}{4/33 \cdot 1/2 + 1 \cdot 1/2} = \frac{4}{37}.$$

$$\boxed{P(A \mid U_1) = \frac{6}{11} \cdot \frac{5}{10} \cdot \frac{4}{9} = \frac{4}{33}}$$

(5) Let $A_i = \{6^{th} \text{ sum equals } i\}$, $i = 2, 3, \dots, 12$. Let B be the event that the sum that appears in 6^{th} roll has not appeared before. We have

$$P(B) = \sum_{i=2}^{12} P(B \mid A_i) P(A_i)$$

To compute $P(A_i)$, consider the 36 possible outcomes



$P(A_2) = P(A_{12}) = \frac{1}{36}$

$P(A_3) = P(A_{11}) = \frac{2}{36}$

$P(A_4) = P(A_{10}) = \frac{3}{36}$

$P(A_5) = P(A_9) = \frac{4}{36}$

$P(A_6) = P(A_8) = \frac{5}{36}$

$P(A_7) = \frac{6}{36}$

We obtain
$$P(B) = 2\left[\left(\frac{35}{36}\right)^5 \frac{1}{36} + \left(\frac{34}{36}\right)^5 \frac{2}{36} + \left(\frac{33}{36}\right)^5 \frac{3}{36} + \left(\frac{32}{36}\right)^5 \frac{4}{36} + \left(\frac{31}{36}\right)^5 \frac{5}{36}\right] + \left(\frac{30}{36}\right)^5 \frac{6}{36} \simeq 0.56$$

$BC = \{\frac{1}{2}\} \cup [\frac{3}{4}, 1]$ $\qquad AB = [\frac{1}{2}, \frac{3}{4}]$

$B \cup C = [\frac{1}{4}, 1]$ $\qquad AC = [\frac{1}{4}, \frac{1}{2}] \cup \{\frac{3}{4}\}$

~~P(A~~ $ABC = \{\frac{1}{2}, \frac{3}{4}\}$

$A(B \cup C) = A = [\frac{1}{4}, \frac{3}{4}]$

(a) $P(AB) = \frac{1}{4}$; $P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Answer: yes

(b) $P(AC) = \frac{1}{4}$; $P(A)P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Answer: yes

(c) $P(ABC) = \frac{1}{4}$; $P(A)P(BC) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. Answer: no

(d) $P(A(B \cup C)) = \frac{1}{2}$; $P(A)P(B \cup C) = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$. Answer: no

ENEE 324 Fall 2016 Problem set 3, Solutions

(1.) Suppose that the tournament has been played, and after that we randomly choose the 2 players in question. There are $\binom{2^n}{2}$ possible pairs of players of which

$$2^{n-1} + 2^{n-2} + \dots + 2 + 1 = 2^n - 1$$

pairs have been actually formed.
In particular, there were $2^{n-1} + 2^{n-2}$ pairs in the 1^{st} and 2^{nd} rounds, so the answer to the first question is

$$\frac{(2^{n-1}+2^{n-2})\cdot 2}{2^n(2^n-1)} = \frac{2^{n-2}(1+2)}{2^{n-1}(2^n-1)} = \frac{3}{2(2^n-1)}$$

Likewise, the other answers are:

$$P(\text{final or semi-final}) = \frac{3}{2^{n-1}(2^n-1)}$$

$$P(\text{never meet}) = \frac{\binom{2^n}{2} - (2^n-1)}{\binom{2^n}{2}} = 1 - \frac{1}{2^{n-1}}$$

(2.) Let B_0, B_1, B_2 be the events that among the 2 removed balls there are 0, 1, 2 blue balls, respectively

$$P(B_0) = \frac{5}{15}\cdot\frac{4}{14} = \frac{2}{21}; \quad P(B_1) = \frac{5}{15}\cdot\frac{10}{14} + \frac{10}{15}\cdot\frac{5}{14} = \frac{10}{21}; \quad P(B_2) = \frac{10}{15}\cdot\frac{9}{14} = \frac{3}{7}$$

Let $A = \{\text{the ball R is blue}\}$. Use the Bayes formula

$$P(B_2|A) = \frac{P(A|B_2)P(B_2)}{P(A|B_0)P(B_0) + P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} =$$

$$= \frac{8/13 \cdot 9/21}{10/13 \cdot 2/21 + 9/13 \cdot 10/21 + 8/13 \cdot 9/21} = \frac{72}{20+90+72} = \frac{36}{91}$$

Problem set 3, Solutions

③ $$P_3 = \frac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}} \approx 0.018; \quad P_4 = \cancel{0.0013; P} = \frac{\binom{6}{4}\binom{43}{2}}{\binom{49}{6}} \approx 0.00027$$

$$P_5 = \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}} \approx 0.000018; \quad P_6 = \frac{1}{\binom{49}{6}} \approx 0.71 \cdot 10^{-7}$$

④ Write the integers $1, 2, \dots, 999999$ in the form

$$\left.\begin{array}{c} 000001 \\ 000002 \\ \vdots \\ 999999 \end{array}\right\} \text{ if 5 is not allowed, there remain } 9^6 \text{ numbers among this set}$$

The probability in question is $\left(1 - \frac{9^6}{10^6}\right) \approx 0.469$

⑤ Let X_i = (no one departs at stop i), $i = 1, 2, 3$

The event in question is $X_1^c X_2^c X_3^c$.

Compute

$$P(X_1 \cup X_2 \cup X_3) = P(X_1) + P(X_2) + P(X_3) - P(X_1X_2) - P(X_2X_3) - P(X_1X_3) + P(X_1X_2X_3)$$

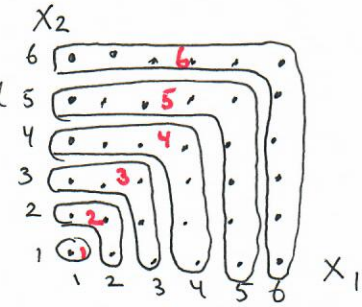
$$= \frac{2^6}{3^6} + \frac{2^6}{3^6} + \frac{2^6}{3^6} - \frac{1}{3^6} - \frac{1}{3^6} - \frac{1}{3^6} + 0 = \frac{7}{27}$$

and find the answer: $P(X_1^c X_2^c X_3^c) = 1 - P(X_1 \cup X_2 \cup X_3) = \frac{20}{27}$.

⑥ Suppose we are placing r marbles in $m+n$ cells; there are $\binom{m+n}{r}$ ways of doing this. In the first m cells there can be $i = 0, 1, \dots, r$ marbles, while the last n cells will contain $n-i$ marbles. For a given i, there are $\binom{m}{i}\binom{n}{r-i}$ placements, so

$$\binom{m+n}{r} = \sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i}$$

Fall 2016 ENEE324 Problem Set 4 solutions



Problem 1 Y_1 takes values 1, 2, 3, 4, 5, 6 according to the figure

PMF

k	1	2	3	4	5	6
$p_{Y_1}(k)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

$$EY_1 = \sum_{k=1}^{6} k\, p_{Y_1}(k) = \frac{1}{36}(1+6+15+28+45+66) = \frac{161}{36}$$

$$EY_1^2 = \sum_{k=1}^{6} k^2 p_{Y_1}(k) = \frac{791}{36}$$

$$Var(Y_1) = EY_1^2 - (EY_1)^2 = \frac{791\cdot 36 - 161^2}{36\cdot 36} = \frac{2555}{1296}$$

Y_2 PMF

k	1	2	3	4	5	6
$p_{Y_2}(k)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

$$EY_2 = \frac{91}{36} \qquad Var(Y_2) = \frac{2555}{1296}$$

$$EY_2^2 = \frac{301}{36}$$

Y_3 PMF

k	2	3	4	5	6	7	8	9	10	11	12
$p_{Y_3}(k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Y_4 PMF

k	−5	−4	−3	−2	−1	0	1	2	3	4	5
$p_{Y_4}(k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Problem 2. The RV X follows a hypergeometric distribution

$$p_X(k) = \frac{\binom{20}{k}\binom{15}{10-k}}{\binom{35}{10}}, \quad k = 0, 1, 2, \dots, 10 \text{ and } p_X(k) = 0 \text{ o/w}$$

Computing the numbers, we obtain

k	0	1	2	3	4	5	6	7	8	9	10
$p_X(k)$	0.000016	0.00054	0.0067	0.040	0.132	0.254	0.288	0.192	0.072	0.014	0.001

Problem 3. Let Y be the dollar amount of win. Range (Y) = 15, 11, 10, 6, 2

PMF

k	15	11	10	6	2
$p_Y(k)$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{6}{15}$	$\frac{3}{15}$

(by counting the number of combinations that produces each amount, divided by all the possible $\binom{6}{2}=15$ pairs)

$$EY = \frac{1}{15}(30+33+10+36+6) = \frac{115}{15} = \frac{23}{3} = 7.666\ldots$$

Thus, if the entry fee is at least \$7.67, the house on average draws a profit.

Problem 4. Let X be the number of tosses till the first time we have seen both H and T, including that time

$p_X(k) = \left(\frac{1}{2}\right)^{k-1}$, $k \geq 2$ because we must have either $\underbrace{HHH\ldots H}_{k-1}T$ or $\underbrace{TTT\ldots T}_{k-1}H$

$$EX = \sum_{i=2}^{\infty} i \left(\frac{1}{2}\right)^{i-1}$$

Let x be a number such that $0<x<1$. We have

$$\sum_{i=2}^{\infty} i x^{i-1} = \sum_{i=1}^{\infty} i x^{i-1} - 1 = \frac{d}{dx}\sum_{i=0}^{\infty} x^i - 1 = \frac{d}{dx}\left(\frac{1}{1-x}\right) - 1 = \frac{1}{(1-x)^2} - 1$$

Substituting $x=\frac{1}{2}$, we find $EX = 3$.

Problem 5 $\binom{6}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 = \frac{160}{729}$

Problem 6. Let X be the RV in question; $p_X(k) = \frac{\lambda^k}{k!}e^{-\lambda}$.

We have $p_X(1) = p_X(4)$, i.e., $\lambda e^{-\lambda} = \frac{\lambda^4}{24}e^{-\lambda}$, i.e. $\lambda = \sqrt[3]{24} \approx 2.88$

$EX = Var(X) = \lambda$ //

ENEE324 (Fall 2016) HW5 Solutions

① First, compute $P(X \text{ even}) = \frac{1}{16}\left[\binom{4}{0}+\binom{4}{2}+\binom{4}{4}\right] = \frac{8}{16} = \frac{1}{2}$

$p_X(k|X \text{ even}) = 0$ for $k=1,3$

$$p_X(k|X \text{ even}) = \frac{1}{P(X \text{ even})} p_X(k) = 2p_X(k) = \begin{cases} 1/8 & k=0,4 \\ 3/4 & k=2 \end{cases}$$

② Let $p_X(k), k=0,1,\dots$ be the Poisson (λ) pmf. We have

$$\sum_{\substack{k\ge 0\\ k \text{ even}}} p_X(k) + \sum_{\substack{k\ge 1\\ k \text{ odd}}} p_X(k) = 1$$

$$\sum_{\substack{k\ge 0\\ k \text{ even}}} p_X(k) - \sum_{\substack{k\ge 1\\ k \text{ odd}}} p_X(k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{(-\lambda)^k}{k!} = e^{-2\lambda}$$

Adding these equalities, we obtain $\sum_{\substack{k\ge 0\\ k \text{ even}}} p_X(k) = \frac{1}{2}(1+e^{-2\lambda})$

③ $\int_1^3 dx/x^2 = \frac{2}{3}$, so by normalization $c = 3/2$

$EX = \frac{3}{2}\int_1^3 \frac{dx}{x} = \frac{3}{2}\ln 3$; $EX^2 = \frac{3}{2}\cdot 2 = 3$; $\operatorname{Var} X = 3 - \frac{9}{4}\ln 3 \approx 0.29$; $2\sigma_X \cong 1.06$

Therefore $P(|X-EX| < 2\sigma_X) \cong P(|X-1.65| < 1.06) = P(0.59 < X < 2.71) \cong 0.95$

④ Again by normalization

$$c\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = c\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d(\sin\theta)}{\sqrt{1-\sin^2\theta}} = c\pi, \text{ so } c = \frac{1}{\pi} \text{ and } f_X(x) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}}, & -1<x<1 \\ 0 & \text{o/w} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x<-1 \\ \int_{-1}^{x} dt/(\pi\sqrt{1-t^2}) = \int_{-\pi/2}^{\arcsin x} d\theta/\pi = \frac{1}{2}+\frac{1}{\pi}\arcsin x, & -1\le x<1 \\ 1, & x\ge 1 \end{cases}$$

⑤ $F_Y(t) = P(X^2 \le t) = P(-\sqrt{t} < X < \sqrt{t}) = \frac{\sqrt{t}}{2}$, $0\le t<4$; $F_Y(t)=0$, $t<0$; $F_Y(t)=1$, $t\ge 4$

$f_Y(t) = F_Y'(t) = \frac{1}{4\sqrt{t}}$, $0<t<4$ and 0 o/w

$F_Z(t) = P(X^3 \le t) = P(X \le \sqrt[3]{t}) = \frac{\sqrt[3]{t}+2}{4}$, $-8 \le t < 8$; $F_Z(t) = 0$, $t<-8$, $F_Z(t)=1$, $t\ge 8$

$f_Z(t) = F_Z'(t) = \frac{1}{12t^{2/3}}$, $-8<t<8$ and 0 o/w

① We find

$$F_X(x) = \begin{cases} 0 & x<0 \\ \frac{4}{\pi}x & 0 \le x < \pi/4 \\ 1 & x \ge \pi/4 \end{cases} \qquad f_X(x) = \frac{4}{\pi} \text{ if } 0 \le x \le \pi/4 \text{ ; } 0 \text{ o/w}$$

$$E(\cos 2X) = \int_0^{\pi/4} \frac{4}{\pi}\cos 2x\,dx = \frac{2}{\pi}\sin 2x \Big|_0^{\pi/4} = \frac{2}{\pi}$$

$$E(\cos^2 X) = E\left(\tfrac{1}{2}(1+\cos 2X)\right) = \frac{1}{2} + \frac{1}{\pi}$$

② $$F_Y(x) = P(X \le \tan x) = \int_{-\infty}^{\tan x} \frac{du}{\pi(1+u^2)} = \frac{1}{\pi}\arctan u \Big|_{-\infty}^{\tan x} = \frac{x}{\pi} + \frac{1}{2}$$

$$f_Y(x) = \frac{1}{\pi} \text{ if } -\frac{\pi}{2} < x < \frac{\pi}{2} \text{ and } 0 \text{ o/w}$$

③ Write $f_X(x)$ as a PDF of a normal RV:

$$f_X(x) = \frac{1}{\sqrt{(1/2)\pi}}\, e^{-\frac{(x-1)^2}{1/2}}$$ This shows that $EX = 1$, $\operatorname{Var} X = \frac{1}{4}$

④ $P(Y \le X) = \frac{1}{2}$; $P(X^2+Y^2 \le 1) = \frac{\pi}{4}$ by calculating the areas of the regions $Y \le X$ and $X^2+Y^2 \le 1$.

⑤ To find the marginal PDFs, compute

$$f_X(x) = \int_0^x 8xy\,dy = 4x^3 \quad 0 \le x \le 1 \text{ ; } \quad f_Y(y) = \int_y^1 8xy\,dy = 4y(1-y^2) \quad 0 \le y \le 1$$

$$f_X(x)f_Y(y) = 16x^3y(1-y^2) \text{ ; } f_{XY}(xy) = 8xy$$, so X and Y are not independent.

⑥ Let $Z = \max(X,Y)$

$$F_Z(z) = P(\max(X,Y) \le z) = P(X \le z \cap Y \le z) \underset{\text{by independence}}{=} P(X \le z)P(Y \le z) = (1-e^{-\lambda z})^2, \quad z \ge 0$$

$$f_Z(z) = F_Z'(z) = 2\lambda e^{-\lambda z}(1-e^{-\lambda z}) = 2\lambda e^{-\lambda z} - 2\lambda e^{-2\lambda z}, \quad z \ge 0$$

$$EZ = 2\lambda\int_0^\infty (z e^{-\lambda z} - z e^{-2\lambda z})\,dz = 2\lambda\left[-\frac{1}{\lambda} z e^{-\lambda z}\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda z}dz + \frac{1}{2\lambda} z e^{-2\lambda z}\Big|_0^\infty - \frac{1}{2\lambda}\int_0^\infty e^{-2\lambda z}dz\right]$$

$$= \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda} //$$

Let us find $f_U(u)$, where $U = \frac{X}{X+Y}$. Put $V = X$

$$\text{Range}(V) = (0, \infty], \quad \text{Range}(U) = (0,1)$$

We have $x = v$, $y = \frac{v}{u} - v$, $J = \begin{vmatrix} 0 & 1 \\ -\frac{v}{u^2} & \frac{1}{u}-1 \end{vmatrix} = \frac{v}{u^2}$

Since $f_{XY}(x,y) = \lambda^2 e^{-\lambda(x+y)}$, $0<x,y<\infty$, we obtain

$$f_{UV}(u,v) = \lambda^2 \frac{v}{u^2} e^{-\lambda(v+\frac{v}{u}-v)} = \lambda^2\frac{v}{u^2} e^{-\lambda\frac{v}{u}}, \quad 0<v<\infty,\ 0<u<1$$

$$f_U(u) = \int_0^\infty f_{UV}(u,v)\,dv = \frac{\lambda^2}{u^2}\int_0^\infty v e^{-\frac{\lambda}{u}v}dv = \frac{u}{\lambda}\int_0^\infty e^{-\frac{\lambda}{u}v} dv = 1, \quad 0<u<1$$

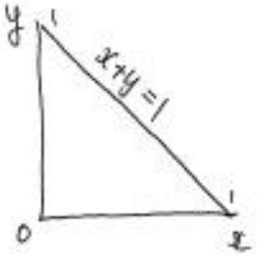
$\boxed{U \sim \text{unif}[0,1]}$

(7) $P(X<1|Y=3) = \int_0^1 f_{X|Y}(x|3)\,dx = \int_0^1 \frac{x+3}{4}e^{-x}dx = \frac14\left[\int_0^1 xe^{-x}dx + 3\int_0^1 e^{-x}dx\right]$

$$= -\frac14 xe^{-x}\Big|_0^1 + \frac14\int_0^1 e^{-x}dx - \frac{3}{4}e^{-x}\Big|_0^1 = -\frac{1}{4e} - \frac{1}{4e} + \frac14 - \frac{3}{4e} + \frac34 = \cancel{\frac{13}{4} - \frac{14}{4e} = \frac{13}{4} - \frac{7}{2e}}$$

$$= 1 - \frac{5}{4e}$$

(8)



$f_{XY}(x,y) = 2$, $x>0$, $y>0$, $x+y<1$ and 0 o/w

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2}{\int_0^{1-y}2\,dx} = \frac{1}{1-y}, \quad 0<x<1-y$$

$$E(X|Y=y) = \frac{1}{1-y}\int_0^{1-y} x\,dx = \frac{1-y}{2}.$$

(9) We have

$x = r\cos\theta$
$y = r\sin\theta$ $\quad J = \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix} = r$

$$f_{R\Theta}(r,\theta) = r f_{XY}(r\cos\theta, r\sin\theta) = \frac{r}{2\pi}e^{-r^2/2}, \quad 0<\theta<2\pi,\ r>0$$

$$f_R(r) = \int_0^{2\pi} f_{R\Theta}(r,\theta)\,d\theta = re^{-r^2/2}, \quad r>0$$

$$f_\Theta(\theta) = \int_0^\infty \frac{re^{-r^2/2}}{2\pi}\,dr = \frac{1}{2\pi}, \quad 0<\theta<2\pi$$

$f_{R\Theta}(r,\theta) = f_R(r)\, f_\Theta(\theta)$, so $R \perp\!\!\!\perp \Theta$.

(1) Let $X_i = \begin{cases} 1 & i^{th} \text{ roll is } 2 \\ 0 & i^{th} \text{ roll is anything else} \end{cases}$; $Y_i = \begin{cases} 1 & i^{th} \text{ roll is } 3 \\ 0 & i^{th} \text{ roll is anything else} \end{cases}$, $i = 1, 2, \ldots, n$

We have $X = \sum_{i=1}^{n} X_i$; $Y = \sum_{i=1}^{n} Y_i$

$$\text{cov}(X,Y) = \text{cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right) = \sum_{i,j=1}^{n} \overbrace{\left(E(X_iY_j) - E(X_i)E(Y_j)\right)}^{\text{cov}(X_i,Y_j)}$$

If $i \neq j$, then X_i and Y_j are independent, so $\text{cov}(X_i, Y_j) = 0$. Continue as follows:

$$= \sum_{i=1}^{n}\left(EX_iY_i - EX_iEY_i\right) = \sum_{i=1}^{n}\left(0 - \frac{1}{6}\cdot\frac{1}{6}\right) = -\frac{n}{36}$$

Note that if X is greater than Y is likely to be smaller, so X and Y are negatively correlated ($\text{cov}(X,Y) < 0$).

(2) We have $f_X(x) = 1$, $0 < x < 1$ and 0 o/w. $EX = \frac{1}{2}$, $EX^2 = \frac{1}{3}$, $EX^4 = \frac{1}{5}$, $EX^3 = \frac{1}{4}$

$$\sigma_X^2 = EX^2 - (EX)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}; \quad \sigma_Y^2 = EY^2 - (EY)^2 = EX^4 - (EX^2)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$$

$$\text{cov}(X,Y) = E(X\cdot X^2) - EX\,EX^2 = \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{12}$$

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{1/12}{\sqrt{\frac{1}{12}\cdot\frac{4}{45}}} = \frac{2\sqrt{3}\cdot 3\sqrt{5}}{12\cdot 2} = \frac{\sqrt{15}}{4}$$

(3) (a) ~~Find~~ $E(X_i | X_{i-1} = m) = \frac{m}{w+b}\cdot m + \frac{w+b-m}{w+b}(m+1) = \frac{m}{w+b}m + \left(1 - \frac{m}{w+b}\right)(m+1)$

$$= 1 + \left(1 - \frac{1}{w+b}\right)m, \quad i \geq 1$$

Thus $E(X_i | X_{i-1}) = 1 + \left(1 - \frac{1}{w+b}\right)X_{i-1}$

Taking the expectation on both sides, we obtain

$$E\left(E(X_i|X_{i-1})\right) = EX_i = 1 + \left(1 - \frac{1}{w+b}\right)EX_{i-1}, \quad i \geq 1$$

see next page

(4) From the formulas provided, we observe that the conditional PDF of Y conditioned on $X=3.6$ is Gaussian with mean

$$\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x-\mu_X) = 2.5 + 0.4\,\frac{0.4}{0.5}(3.6-3) = 2.692$$

and variance

$$\sigma = \sigma_Y\sqrt{1-\rho^2} = 0.4\sqrt{0.84} \approx 0.37$$

Then, denoting $Z \sim \mathcal{N}(0,1)$ a standard normal RV, we obtain

$$P(Y \geq 3 \mid X=3.6) = P\left(\frac{Y-2.692}{0.37} \geq \frac{3-2.692}{0.37} \,\Big|\, X=3.6\right) = P(Z \geq 0.832)$$

$$= 1 - \Phi(0.832) \approx 1 - 0.797 = 0.203$$

(3) Let X_i = # blue balls after i trials, $i=1,2,\ldots,n$. Suppose that $X_i = m$, then

$$E(X_i \mid X_{i-1}=m) = \frac{m}{B}m + \frac{B-m}{B}(m+1) = 1 + \left(1-\frac{1}{B}\right)m, \quad i \geq 1$$ (if $i=1$, then the condition is $X_0 = b$)

In particular $EX_1 = 1 + \left(1-\frac{1}{B}\right)b = B - r\left(1-\frac{1}{B}\right)$

Generally we find ~~(using~~

$$E(X_i \mid X_{i-1}) = 1 + \left(1-\frac{1}{B}\right)X_{i-1}, \quad i \geq 1$$

Compute the expectation of this relation, recalling that $E\big(E(X_i \mid X_{i-1})\big) = EX_i$:

$$\text{(A)} \qquad EX_i = 1 + \left(1-\frac{1}{B}\right)EX_{i-1}, \quad i \geq 1$$

Now take $i=n$ and use (A) repeatedly $n-1$ times:

$$EX_n = 1 + \left(1-\frac{1}{B}\right)EX_{n-1} = 1 + \left(1-\frac{1}{B}\right)\left(1 + \left(1-\frac{1}{B}\right)EX_{n-2}\right) = 1 + \left(1-\frac{1}{B}\right) + \left(1-\frac{1}{B}\right)^2 EX_{n-2}$$

$$= 1 + \left(1-\frac{1}{B}\right) + \left(1-\frac{1}{B}\right)^2\left(1 + \left(1-\frac{1}{B}\right)EX_{n-3}\right) = 1 + \left(1-\frac{1}{B}\right) + \left(1-\frac{1}{B}\right)^2 + \left(1-\frac{1}{B}\right)^3 EX_{n-3}$$

$$= \ldots = \underbrace{1 + \left(1-\frac{1}{B}\right) + \left(1-\frac{1}{B}\right)^2 + \ldots + \left(1-\frac{1}{B}\right)^{n-2}}_{\text{geom. summation}} + \left(1-\frac{1}{B}\right)^{n-1} EX_1$$

$$\overset{**}{=} B - B\left(1-\frac{1}{B}\right)^{n-1} + \left(1-\frac{1}{B}\right)^{n-1}\left(B - r\left(1-\frac{1}{B}\right)\right)$$

$$= B - r\left(1-\frac{1}{B}\right)^n \;//$$

(5) We have

$$u = x+y \qquad \text{so} \qquad x = \tfrac{1}{2}(u+v) \qquad J = \tfrac{1}{2}$$
$$v = x-y \qquad\qquad y = \tfrac{1}{2}(u-v)$$

$$f_{UV}(u,v) = f_{XY}(x+y,\, x-y)\cdot J$$

and we conclude that the pair U,V follows the bivariate normal distribution

Further $\quad \mathrm{cov}(U,V) = \mathrm{cov}(X+Y, X-Y) = \mathrm{var}(X) - \mathrm{var}(Y) = 0$

so U and V are uncorrelated. As shown in class, two uncorrelated Gaussian RVs are independent, so the answer is Yes. (see also p.256 of the textbook)

(6) (a) Since $\quad M_X(s) = \sum p_X(k)\, e^{sk}$

we see that $p_X(k) \neq 0$ if $k = 1,3,4,7$. By the uniqueness of transform property we find the PMF

$p_X(k)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{1}{10}$
k	1	3	4	7

and from this compute $P(X \leq 5) = \frac{9}{10}$ //

(b) Given $M_X(s) = e^{2s^2}$

The transform of a Gaussian $N(\mu, \sigma^2)$ is $e^{\mu s + \frac{\sigma^2 s^2}{2}}$, so using the uniqueness of transforms property, we conclude that our RV is Gaussian with mean $\mu = 0$ and $\frac{\sigma^2}{2} = 2$, i.e., $\sigma = 2$.

Now find

$$P(0 < X < 1) = P\left(0 < \frac{X-\mu}{\sigma} < \frac{1-\mu}{\sigma}\right) = P\left(0 < Z < \tfrac{1}{2}\right), \text{ where } Z \sim N(0,1)$$

$$= \Phi\left(\tfrac{1}{2}\right) - \Phi(0) \cong 0.69 - 0.5 = 0.19 \,//$$

Problem 1. From the table on p. 239,

$X \sim \text{Poisson}(2) \qquad p_X(i) = e^{-2}\frac{2^i}{i!}, \; i=0,1,2,\ldots \qquad EX=2$

$Y \sim \text{Binom}(10, \frac{3}{4}) \quad P_Y(0) = \frac{1}{4^{10}}; \; P_Y(1) = 10\cdot\frac{3}{4}\frac{1}{4^9}; \; P_Y(2) = 45\left(\frac{3}{4}\right)^2\frac{1}{4^8}$

The below calculation is for X+Y=2; the calculation for X+Y=3 is analogous

$$P(X+Y=2) = P(X=0,Y=2) + P(X=1,Y=1) + P(X=2,Y=0)$$

$$= \frac{1}{4^{10}} e^{-2} \cdot \frac{4}{2} + 10\cdot\frac{3}{4^{10}} e^{-2}\cdot 2 + 45\frac{9}{4^{10}}e^{-2} = \frac{e^{-2}}{4^{10}}(2+60+405) = 467\frac{e^{-2}}{4^{10}}$$

$$P(XY=0) = P(X=0 \text{ or } Y=0) = P(X=0,Y=0) + P(X=0,Y\geq 1) + P(X\geq 1, Y=0)$$

$$= P(X=0) + P(Y=0) - P(X=0,Y=0) = e^{-2} + 4^{-10} - e^{-2}\cdot 4^{-10} = e^{-2}(1-4^{-10}) + 4^{-10}$$

$$EXY = EX\,EY = 2\cdot\left(10\cdot\frac{3}{4}\right) = 15$$

Problem 2. Let $Y = \sum_{i=1}^{20} X_i$; we have $EX_i = 1$; $Var(X_i) = 1$; $EY = 20$

(a) $P(Y>25) \leq \frac{EY}{25} = \frac{4}{5}$ using the Markov inequality

(b) using the CLT,

$$P\left(\frac{Y-20}{\sqrt{20}} > \frac{25-20}{\sqrt{20}}\right) \approx P\left(\frac{Y-20}{\sqrt{20}} > 1.12\right) \approx 1 - \Phi(1.12) \approx 0.1314$$

Problem 3. Let $X = \sum_{i=1}^{10000} X_i$; $\mu = 240$, $n\mu = 10^4 \cdot 240 = 2.4\cdot 10^6$; $\sigma\sqrt{n} = 8\cdot 10^4$

$$P\left(\frac{X-2.4\cdot 10^6}{8\cdot 10^4} > \frac{2.7\cdot 10^6 - 2.4\cdot 10^6}{8\cdot 10^4}\right) \simeq P(Z>3.75), \text{ where } Z\sim\mathcal{N}(0,1)$$

So the answer is $1-\Phi(3.75) \approx 1 - 0.999912 = 0.000088$

Problem 4. Choose $\varepsilon > 0$. ~~For~~ It looks as though $Y_i, i=1,2,..$ converge to 0, so let us check this possibility.

$$P(|Y_i - 0| > \varepsilon) = 1 - P(|Y_i| \leq \varepsilon) = 1 - P(X \leq i) = 1 - \int_0^i f_X(x)\,dx$$

As $i \to \infty$, the last integral approaches 1, so

$$\lim_{i\to\infty} P(|Y_i| > \varepsilon) = 0,$$

which shows that the sequence Y_i converges to 0 in probability.

Problem 5

(a) $P(N(\tfrac{1}{3})=2) = (\tfrac{1}{3}\lambda)^2 \dfrac{e^{-\lambda/3}}{2} = \dfrac{1}{18}\lambda^2 e^{-\lambda/3}$

$$P(N(\tfrac{1}{3})=2 \mid N(1)=2) = P(N(\tfrac{1}{3})=2)\,P(N(\tfrac{2}{3})=0) / P(N(1)=2) = \frac{(1/18)\lambda^2 e^{-\lambda/3}\cdot e^{-\frac{2\lambda}{3}}}{\lambda^2 e^{-\lambda}/2!} = \frac{1}{9}$$

(b) $\cancel{1-P(N(\tfrac{1}{3})=0) = 1-e^{-\lambda/3}}$

The event that at least one packet arrived within $\frac{1}{3}$ time is complementary to the event that none of them have arrived within that time.

$$P(N(\tfrac{1}{3})\neq 0 \mid N(1)=2) = 1-P(N(\tfrac{1}{3})=0 \mid N(1)=2) = 1-\frac{P(N(\tfrac{1}{3})=0)\,P(N(\tfrac{2}{3})=2)}{P(N(1)=2)}$$

$$= 1-\frac{e^{-\lambda/3}\cdot(2\lambda/3)^2 e^{-2\lambda/3}}{2\cdot\lambda^2 e^{-\lambda}/2} = 1-\frac{4}{9} = \frac{5}{9}$$

(c) We need to find $P(N(s)=i \mid N(t)=n),\ 0\leq i\leq n$

$$P(N(s)=i \mid N(t)=n) = \frac{P(N(s)=i,\ N(t)=n)}{P(N(t)=n)} = \frac{P(N(s)=i,\ N(t-s)=n-i)}{P(N(t)=n)}$$

$$= \frac{P(N(s)=i)\,P(N(t-s)=n-i)}{P(N(t)=n)} = \frac{n!}{i!\,(n-i)!}\,\frac{e^{-\lambda s}(\lambda s)^i\, e^{-\lambda(t-s)}(\lambda(t-s))^{n-i}}{e^{-\lambda t}(\lambda t)^n}$$

$$= \frac{n!}{i!\,(n-i)!}\left(\frac{s}{t}\right)^i\left(1-\frac{s}{t}\right)^{n-i} \sim \text{Binom}\left(n, \frac{s}{t}\right)$$

Problem 6.

The error process is Poisson with $\lambda = \frac{1}{60}$. Let $N(t)$ be the number of errors in t trasmissions, so $N(t) \sim \text{Poisson}(\lambda t)$.

In 10 sec. there are 70 transmissions, and we would like to find $P(N(70)>1)$:

$$P(N(70)>1) = 1-P(N(70)=0)-P(N(70)=1) = 1-e^{-70/60}-\frac{70}{60}e^{-70/60} = 1-\frac{13}{6}e^{-7/6}$$

≈ 0.325

ENEE 324 Problem set 9

Prob. 1. $R(n) = P^n$, where
$$P = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

Computing $r_{11}(4)$, we find $r_{11}(4) = 3/8$

$$r_{12}(5) + r_{16}(5) = \frac{11}{32} + \frac{11}{32} = \frac{11}{16}$$

Prob. 2. By inspection, 2 is transient, and $\{1,4\}$ and $\{3,5\}$ form disjoint recurrent classes.

Prob. 3. The chain has 3 states, 1, 2 and 3, corresponding to the number of the player that is currently drawing the card. The transitions are given by the following matrix:

$$P = \begin{bmatrix} 48/52 & 4/52 & 0 \\ 0 & 39/52 & 13/52 \\ 12/52 & 0 & 40/52 \end{bmatrix}$$

Solving the system $\pi P = \pi$, where $\pi = (\pi_1, \pi_2, \pi_3)$, we obtain
$$\pi_1 = \frac{39}{64}; \quad \pi_2 = \frac{12}{64}; \quad \pi_3 = \frac{13}{64}$$

Prob. 4. Let $x_i = \text{Prob}(\text{The chain is absorbed into } 4 \mid \text{start from } i)$, $i = 1, 2, \ldots, 7$.

Using the total probability theorem, we obtain

$$\left.\begin{aligned} x_1 &= 0.3x_1 + 0.7x_2 \\ x_2 &= 0.3x_1 + 0.2x_2 + 0.5x_3 \\ x_3 &= 0.6x_4 + 0.4x_5 \\ x_4 &= 1 \\ x_5 &= (1)\cdot x_3 \\ x_6 &= 0.1x_1 + 0.3x_2 + 0.1x_3 + 0.2x_5 + 0.6x_6 + 0.1x_7 \\ x_7 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} & x_1 = x_2 = x_3 = x_4 = x_5 = 1 \\ & x_6 = 0.875 \\ & x_7 = 0 \end{aligned}$$

Answer: 0.875

Prob. 5. Use the total expectation theorem.

Average wait till 1

$t_2 = 1 + 0.15t_2 + 0.53t_3$

$t_3 = 1 + 0.13t_2 + 0.27t_3$

$t_2 = 2.28$, $t_3 = 1.77$

$\boxed{\text{Ans: } 1.67}$

found as $\frac{1}{3}(1 + t_2 + t_3)$

till 2

$t_1 = 1 + 0.2t_1 + 0.5t_3$

$t_3 = 1 + 0.6t_1 + 0.27t_3$

$t_1 = 4.33$; $t_3 = 4.92$

$\boxed{\text{Ans.: } 3.38}$

$\frac{1}{3}(t_1 + 1 + t_3)$

till 3

$t_1 = 1 + 0.2t_1 + 0.3t_2$

$t_2 = 1 + 0.32t_1 + 0.15t_2$

$t_1 = 1.97$; $t_2 = 1.92$

$\boxed{\text{Ans.: } 1.61}$

$\frac{1}{3}(t_1 + t_2 + 1)$

(6) Let i = # of chargers at the current location (home or office), $i = 0, 1, \dots, r$. Then

$p_{i,i+1} = 1-p$; $p_{ii} = p$, $i = 1, 2, \dots, r$, and $p_{0r} = 1$.

$$P = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & p & 1-p \\ 0 & 0 & 0 & \cdots & p & 1-p & 0 \\ \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & p & 1-p & & 0 & 0 & 0 \\ p & 1-p & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

Balance equations

$$\pi_0 = p\pi_r$$
$$\pi_1 = p\pi_{r-1} + (1-p)\pi_r \rightarrow$$
$$\pi_2 = p\pi_{r-2} + (1-p)\pi_{r-1}$$
$$\vdots$$
$$\pi_{r-1} = p\pi_1 + (1-p)\pi_2$$
$$\pi_r = \pi_0 + (1-p)\pi_1$$

using the first and the last equations, we obtain $\pi_r = \pi_1$

then ~~next to the last~~ the second one gives $\pi_{r-1} = \pi_r$, and generally

$$\pi_1 = \pi_2 = \dots = \pi_{r-1} = \pi_r = \frac{1}{p}\pi_0$$

(a) So

$$1 = \pi_0 + \pi_1 + \dots + \pi_r = \pi_r(p + \overbrace{1 + 1 + \dots + 1}^{r}) \Rightarrow \boxed{\pi_1 = \dots = \pi_r = \frac{1}{p+r};\ \pi_0 = \frac{p}{p+r}}$$

(b) for $r = 3$, the proportion of time in state 0 is $\pi_0 = \dfrac{p}{p+2}$

(c) $\dfrac{p}{p+2}$ is maximal when $p = 1$.