

## ENEE324-03: Engineering Probability

Midterm Examination: Problems and solutions

Oct. 20, 2104

**Problem 1.** There are two bins with black and white balls. The first bin contains 8 white and 12 black balls. The second bin contains 4 white and 6 black balls. A ball is taken at random from the first bin and put into the second bin. Then a ball is taken at random from the second bin. What is the probability that this ball is white?

**Solution.** Let  $A$  be the event that the ball taken from the second bin is white. Let  $B$  be the event that the ball taken from the first bin is black and  $W$  be the event that it is white. We have

$$\mathbf{P}(B) = \frac{12}{8+12} = 0.6, \quad \mathbf{P}(W) = \frac{8}{8+12} = 0.4.$$

Next,

$$\mathbf{P}(A|B) = \frac{4}{11}, \quad \mathbf{P}(A|W) = \frac{5}{11}.$$

By the total probability formula,

$$\mathbf{P}(A) = \mathbf{P}(A|B)\mathbf{P}(B) + \mathbf{P}(A|W)\mathbf{P}(W) = \frac{4}{11} \cdot 0.6 + \frac{5}{11} \cdot 0.4 = \frac{4.4}{11} = 0.4$$

**Problem 2.** Two coins are tossed simultaneously until one of them (does not matter which) comes up heads and the remaining one comes up tails. The probability that the first coin comes up heads is  $p$  and the probability that the second coin comes up heads is  $q$ .

- (a) Find the PMF, the expected value, and the variance of the number of tosses.
- (b) In the same experiment, instead of stopping after the first occurrence of HT, we always make  $N$  tosses and then stop. Let  $X$  random number of the outcomes HH. Find the PMF of  $X$ ,  $E(X)$ ,  $\text{Var}(X)$ .

**Solution.** (a) Let  $X$  be the r.v. corresponding to the random number of tosses. The probability of the needed outcome in one toss of the coins is  $p_0 = p(1-q) + q(1-p)$ . Then  $X \sim \text{geom}(p_0)$ , so  $p_X(k) = (1-p_0)^{k-1}p_0$ ,  $E[X] = 1/p_0$ ,  $\text{var}(X) = \frac{1-p_0}{p_0^2}$ .

- (b) This time  $X$  is a binomial r.v. with the probability of success in one trial  $\mathbf{P}(HH) = pq$ . So

$$p_X(k) = \binom{N}{k} (pq)^k (1-pq)^{N-k}, \quad k = 0, 1, \dots, N,$$

$$E[X] = Npq, \quad \text{Var}(X) = Npq(1-pq).$$

**Problem 3.** Consider a sequence of independent Bernoulli trials with  $\mathbf{P}(\text{success}) = 0.8$ . Let  $X$  be the trial number of the first success and  $Y$  the trial number of the second success.

- (a) Find  $p_{Y|X}(y|x)$  for  $1 \leq x \leq 4, 1 \leq y \leq 4$ .
- (b) Find  $p_{X,Y}(x,y)$  for the same range of  $x$  and  $y$ .

**Solution.** (a) Let us find  $p_{Y|X}(j|i)$ . Clearly,  $Y > X$ , so  $p_{Y|X}(j|i) = 0$  if  $j \leq i$ . Otherwise, we have

$$p_{Y|X}(2|1) = 0.8, p_{Y|X}(3|1) = 0.2 \cdot 0.8, p_{Y|X}(4|1) = (0.2)^2 \cdot 0.8,$$

and so on. We therefore obtain for  $p_{Y|X}$  in the range of interest

$y$					
4	0.032	0.16	0.8	0	
3	0.16	0.8	0	0	
2	0.8	0	0	0	
1	0	0	0	0	
	1	2	3	4	$x$

(b) We know that  $p_X(k) = 0.2^{k-1}0.8$ ,  $k = 1, 2, \dots$ , So

$x$	1	2	3
$p_X(k)$	0.8	0.16	0.032

Compute  $p_{X,Y}(i, j) = p_{Y|X}(j|i)p_X(i)$ . We obtain

$y$					
4	0.0256	0.0256	0.0256	0	
3	0.128	0.128	0	0	
2	0.64	0	0	0	
1	0	0	0	0	
	1	2	3	4	$x$

**Problem 4.**

Below  $A, B, C$  denote events (subsets of the universal set  $\Omega$ ). For each of the following relations, determine if it is correct or false.

$$\begin{aligned} A \cup B &= (A \setminus AB) \cup B \\ AB^cC &\subset A \cup B \\ ABC &= AB(C \cup B) \end{aligned}$$

**Solution.** The first two relations are correct, the third one is wrong. The first is correct, by drawing a Venn diagram. The second is also correct because on the left of the  $\subset$  sign it refers to a subset of the set  $A$  and on the right it refers to all of the set  $A$  (and some other elements). Therefore any  $x$  that is contained in the left part, is also contained in the right part.

The third relation is wrong because on the left of the  $=$  sign it is required that  $x \in C$ . At the same time, the right part refers to elements  $x$  that are in  $C$  or  $B$  and in  $A$ , therefore any element  $x$  such that  $x \in A, x \in B, x \notin C$  is contained in the right set and is not contained in the left set.

**Problem 5** Two fair dice are rolled once. Let  $X$  be the outcome for the first die and  $Z$  the maximum of the results for the first and the second die.

Find the pmf  $p_Z(k)$ ,  $k = 1, 2, 3, 4, 5, 6$ . Find  $EX, Var(X), EZ$ .

**Solution.** Since  $X$  is a uniform random variable, we have  $\mathbf{E}[X] = 3.5$ ,  $\text{var}(X) = \frac{(6-1)(6-1+2)}{12} = \frac{35}{12}$ .

Find  $p_Z(k)$ . For this, let  $Y$  be the random value of the second die, and note that  $Z = k$  if  $(X, Y) = (1, k), (2, k), \dots, (k-1, k), (k, k), (k, k-1), \dots, (k, 1)$ , altogether  $2k-1$  possibilities, and the probability of each is  $1/36$ . We obtain  $p_Z(k) = (2k-1)/36$ ,  $k = 1, \dots, 6$ , i.e.,  $1/36, 3/36, 5/36, 7/36, 9/36, 11/36$ . Then

$$EZ = \frac{1}{36}(1 + 2 \cdot 3 + 3 \cdot 5 + 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11) = \frac{1}{36}(1 + 6 + 15 + 28 + 45 + 66) = \frac{161}{36}.$$