ENEE324-03: Engineering Probability

Midterm Examination #2 **Nov. 17, 2014**

Problems 1,3,4 are 10 pts each; Problem 2 is 15 pts. Instructor: A. Barg

• Please be precise and rigorous in your statemens, and show all your calculations. Please write neatly and legibly.

• This is a closed book exam, but you are allowed up to two 8.5 × 11 pages of notes.

No calculators please!

Problem 1 (10pts)
The time-to-burn of Light Bulb 1 is an exponential RV $X \sim \exp(\lambda)$. The time-to-burn of Light Bulb 2 is also exponential, $Y \sim \exp(\mu)$, and X and Y are independent. Both light bulbs are switched on at time $t = 0$. If Z is the random time till one of the two bulbs burns, what is the distribution of Z ? (hint: model Z as $\min(X, Y)$ and use Venn diagrams to express $P(Z \le z)$ via $F_X(z)$ and $F_Y(z)$.)

Solution: The CDF of an exponential RV with the parameter λ is $F_X(x) = 1-e^{-\lambda x}, 0 \le x \le \infty$.

$$\begin{aligned} F_Z(z) &= P(\{X \le z\} \cup \{Y \le z\}) = P(\{X \le z\}) + P(\{Y \le z\}) - P(\{X \le z\} \cap \{Y \le z\}) \\ &= F_X(z) + F_Y(z) - F_X(z)F_Y(z) = 1 - e^{-\lambda z} + 1 - e^{-\mu z} - (1 - e^{-\lambda z})(1 - e^{-\mu z}) \\ &= 1 - e^{-(\lambda+\mu)z}. \end{aligned}$$

This is the CDF of $\exp(\lambda + \mu)$. Answer $Z \sim \exp(\lambda + \mu)$.

Problem 2 (15 pts)
A random point X is chosen on the segment $[0, 1]$ with uniform distribuion. After that, a random point Y is chosen on the segment $[0, X]$, also with uniform distribution.

(a) Find $f_{Y|X}(y|x), f_{XY}(x, y), f_Y(y)$ (hint: compute the functions in this order). Compute EY using $f_Y(y)$ and the definition of mathematical expectation.

(b) Use the law of iterated expectations to compute EY in a different way than in (a); check that you get the same answer.

(c) If X and Y are random sides of a rectangle in the (x, y) plane, what's the expected area of the rectangle?

Solution: (a) Since given $X = x$, Y is uniform on $[0, x]$, we find

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & \text{if } 0 \le y < x \\ 0 & \text{if } 1 \ge y \ge x \ge 0 \end{cases}$$

Since $f_X(x) = 1$, we obtain

$$f_{XY}(x, y) = f_{Y|X}(y|x)f_X(x) = \begin{cases} \frac{1}{x} & \text{if } 0 \le y < x,\ 0 < x \le 1 \\ 0 & \text{o/w} \end{cases}$$

Now

$$f_Y(y) = \int_y^1 \frac{dx}{x} = -\ln y \text{ (if } 0 < y \le 1); \quad EY = -\int_0^1 y \ln y dy = -\frac{y^2}{2} \ln y\Big|_0^1 + \int_0^1 \frac{ydy}{2} = \frac{1}{4}.$$

(b) $EY = E_X[E[Y|X]] = E[X/2] = 1/4.$

(c) $E[XY] = \int_0^1 \int_0^x xy(\frac{1}{x})dydx = \int_0^1 \frac{x^2}{2}dx = 1/6.$

Problem 3 (10pts)
The joint pdf of RVs X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{1}{16\pi} & \text{if } x^2+y^2 \le 16 \\ 0 & \text{o/w} \end{cases}$$

(a) Find $f_X(x)$, $E[X|Y]$

(b) Let $R = \sqrt{X^2+Y^2}$. Find $f_R(r)$ (hint: the pair X, Y is jointly uniformly distributed in the circle).

(c) Are X and Y independent?

Solution: (a) $f_X(x) = \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} f_{XY}(x,t)dt = \frac{\sqrt{16-x^2}}{8\pi}, -4 \le x \le 4$. By symmetry we have $f_Y(y) = \frac{\sqrt{16-y^2}}{8\pi}$ and $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{2\sqrt{16-y^2}}$, $-\sqrt{16-y^2} \le x \le \sqrt{16-y^2}$. Then for any $y, -4 \le y \le 4$

$$E[X|Y=y] = \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \frac{xdx}{2\sqrt{16-y^2}} = 0$$

and $E[X|Y] = 0$ with probability one (this is of course clear by symmetry without any computations).

(b)

$$F_R(r) = \begin{cases} 0 & \text{if } r \le 0 \\ P(R \le r) = P(X^2+Y^2 \le r^2) = \frac{\pi r^2}{16\pi} = (\frac{r}{4})^2 & \text{if } 0 \le r \le 4 \\ 1 & \text{if } r \ge 4 \end{cases},$$

and $f_R(r) = F_R'(r) = r/8$ if $0 \le r \le 4$ and $= 0$ otherwise.

(c) From the definition and the PDFs computed above it is immediate that X and Y are not independent.

Problem 4 (10pts)
Suppose that two dependent RVs X and Y have standard deviations $\sigma_X = 2, \sigma_Y = 5$ and $\rho_{XY} = -1/3$. Find $\text{cov}(X,Y)$ and $\text{Var}(4X-2Y)$ (hint: correct answers should be numbers only and should not include any letters).

Solution: Since $\rho_{XY} = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$, we obtain $\text{cov}(X,Y) = -\frac{10}{3}$.

$$\begin{aligned} \text{Var}(4X-2Y) &= E[(4X-2Y)^2] - (4EX-2EY)^2 \\ &= 16EX^2 - 16E(XY) + 4EY^2 - 16(EX)^2 + 16EXEY - 4(EY)^2 \\ &= 16\text{Var}(X) + 4\text{Var}(Y) - 16cov(X,Y) = 64 + 100 - \frac{-160}{3} = \frac{652}{3}. \end{aligned}$$