ENEE324-03: Engineering Probability

Final Examination **Dec. 18, 2014**

The exam consists of SIX problems

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No calculators please. Simplify the answers as much as you can. Print your name on your paper!

Problem 1 (10pts)

Consider a random variable (RV) X with the cumulative distribution function (CDF)

$$F_X(x) = \begin{cases} 0, & x < -1 \\ 0.2, & -1 \le x < 0 \\ 0.7, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

(a) What is the probability mass function (PMF) of X? What is the probability density function (PDF) of X?

(b) Consider $Y = |X|$. Find the PMF and CDF of Y; find $E[Y]$.

Problem 2 (10pts)
An RV X has PDF $f_X(x) = x/2$ for $0 \le x \le 2$ and 0 otherwise. X is clipped to Y using the circuit that outputs

$$Y = \begin{cases} 1/2 & 0 \le X \le 1 \\ X, & X > 1. \end{cases}$$

Find $P(Y = 1/2)$. Find the CDF $F_Y(y)$.

Problem 3 (10 pts) RVs X and Y have the joint PDF

$$f_{XY}(x,y) = \begin{cases} x+y, & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = \max(X,Y)$.

(a) Find the range of Z.

(b) Find $F_Z(z)$ and $f_Z(z)$ (make sure to give your answer for all values of z).

Problem 4 (10pts)
A sequence of RVs $X_1, X_2, \dots,$ is formed of independent, identically distributed RVs with finite mean m and positive variance. Define random variables $Y_i = 0.2X_i + 0.8X_{i+1}, i = 1, 2, \dots$.

(a) Are the RVs Y_i independent?

(b) Are the RVs Y_i identically distributed?

(OVER PLEASE)

(c) Let $Z_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Does the sequence $Z_n, n = 1, 2, \ldots$ converge to m in probability?

Problem 5 (10pts)

Consider the Markov chain with states $i = 1, 2, 3$ and transition probabilities given by the matrix

i	1	2	3
1	0.4	0.6	0
2	0.3	0.5	0.2
3	0	0.2	0.8

(a) Find the steady-state distribution $\pi = (\pi_1, \pi_2, \pi_3)$ of the chain.

Assume that $P(X_0 = i) = \pi_i$. We say that a transition $i \to j$ is of type B if $i < j$.

(b) What is the probability that the first transition we observe is of type B?

(c) What is the probability that the first change of state we observe is of type B? (A change of state is a transition $i \to j$ such that $i \neq j$.)

(d) What is the conditional probability that the chain was in state 2 before the first transition that we observe, given that this transition was of type B?

Problem 6 (10pts)

Requests arriving at the input of a server follow a Poisson process with arrival rate $\lambda = 0.5$. Compute the following quantities:

(a) Probability of no arrivals within the first 3 time units.

(b) Let T be the random time of the 1st arrival. Find the probability $P(4 \le T \le 6)$.

(c) Let S be the random time of the second arrival. Find $E(S^2)$.

Solution of Problem 6:

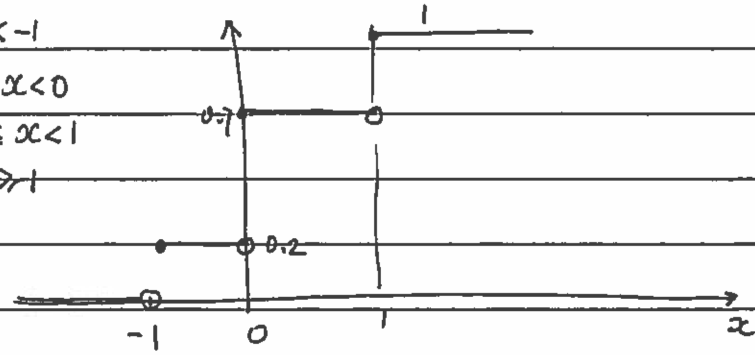
(a) $P(0, 3) = e^{-3\lambda} = e^{-1.5}$.

(b) $P(4 \le T \le 6) = P(0, 4)(1 - P(0, 2)) = e^{-4\lambda}(1 - e^{-2\lambda}) = e^{-2} - e^{-3}$.

(c) We know that $ES = 2/\lambda = 4$, $\text{Var}(S) = 2/\lambda^2 = 8$, so $ES^2 = \text{Var}(S) + (ES)^2 = 24$.

ENEE324 - 2014 Solutions of Final exam

(1) Given $F_X(x) = \begin{cases} 0 & x < -1 \\ 0.2 & -1 \le x < 0 \\ 0.7 & 0 \le x < 1 \\ 1 & x \geq 1 \end{cases}$



~~(a) PMF of X is found by noticing that X is a discrete RV taking values −1, 0, 1 with positive probability.~~

$$p_X(k) = \begin{cases} 0.2 & k = -1 \\ 0.5 & k = 0 \\ 0.3 & k = 1 \\ 0 & o/w \end{cases} \qquad \underline{\text{PDF}} \quad f_X(x) = 0.2\,\delta(x+1) + 0.5\,\delta(x) + 0.3\,\delta(x-1)$$

To compute $p_X(k)$ note that $F_X(x) = P(X \le x)$. So $P(X \leq -1) = 0.2$

~~$P_X(X \leq 0) = 0.7$, $P(X \leq 1) = 1$~~

$p_X(-1) = 0.2$; $p_X(0) = P(X \le 0) - p_X(-1) = 0.5$; $p_X(1) = P(X \leq 1) - 0.7 = 0.3$

(b) $Y = |X|$ $\quad p_Y(1) = p_X(1) + p_X(-1)$

PMF $\quad p_Y(k) = \begin{cases} 0.5 & k = 1 \\ 0.5 & k = 0 \\ 0 & o/w \end{cases}$; $EY = 0.5$; $F_Y(y) = \begin{cases} 0 & y < 0 \\ 0.5 & 0 \le y < 1 \\ 1 & y \geq 1 \end{cases}$

(2) (2) $\quad P(Y = \frac{1}{2}) = P(0 \le X \le 1) = \int_0^1 \frac{x}{2}\,dx = \frac{1}{4}$

$$F_Y(y) = \begin{cases} 0 & y < 1/2 \\ \frac{1}{4} & \frac{1}{2} \le y < 1 \\ y^2/4 & 1 \le y \leq 2 \\ 1 & y \geq 2 \end{cases}$$

Problem 3 : (a) Since $0 \le X \le 1$ and $0 \le Y \le 1$, we find that

$Z = \max(X, Y)$ satisfies $\boxed{0 \le Z \le 1}$

(b) We need to find $F_Z(z) = P(Z \le z) = P(X \le z \cap Y \le z)$

So
$$F_Z(z) = \int_0^z \int_0^z f_{XY}(x,y)\,dx\,dy$$
$$= \int_0^z \int_0^z (x+y)\,dy\,dx = \int_0^z \left(xy + \frac{y^2}{2}\right)\Big|_0^z dx = \int_0^z \left(xz + \frac{z^2}{2}\right) dx = z^3$$

Ans:
$$F_Z(z) = \begin{cases} 0 & z < 0 \\ z^3 & 0 \le z < 1 \\ 1 & z \ge 1 \end{cases} \qquad f_Z(z) = F_Z'(z) = 3z^2, \quad 0 \le z \le 1$$

4. (a) The answer is no; we show this by showing that $E Y_i Y_j \neq E Y_i E Y_j$:

$$E(Y_1 Y_2) = E\left[(0.2X_1 + 0.8X_2)(0.2X_2 + 0.8X_3)\right]$$
$$= E\left[0.04X_1X_2 + 0.16X_1X_3 + 0.16X_2^2 + 0.64X_2X_3\right] = 0.84m^2 + 0.16EX_2^2.$$
$$EY_1 \; EY_2 = m^2$$

If ~~Generally~~ $0.84m^2 + 0.16EX_2^2 = m^2$, then $EX_2^2 = m^2$, but then $\operatorname{Var} X_2 = 0$. This means that this assumption is invalid.

(b) Yes: The distribution of the pair (X_i, X_{i+1}) is the same ~~as~~ for all i.

(c) We have

$$Z_n = \frac{1}{n}\sum_{i=1}^{n} Y_i = 0.2\cdot\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) + 0.8\left(\frac{1}{n}\sum_{i=1}^{n} X_{i+1}\right)$$

Each of the terms in the parentheses converges in prob. to m, so $Z_n \to m$.

5. (a)
$$\left.\begin{aligned} \pi_1 &= 0.4\pi_1 + 0.3\pi_2 \\ \pi_3 &= 0.2\pi_2 + 0.8\pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}\right\}$$

From the 1st line $0.6\pi_1 = 0.3\pi_2$, i.e., $2\pi_1 = \pi_2 = \pi_3$

$\Rightarrow \pi_1 = 0.2; \quad \pi_2 = \pi_3 = 0.4$

(b) The prob. in question $= \pi_1 \cdot p_{12} + \pi_2 \cdot p_{23} = 0.2\cdot 0.6 + 0.4\cdot 0.2 = 0.2$

(c) If $X_0 = 1$, then the chain moves to $i=2$ with prob. 0.6 in one step; $0.4\cdot 0.6$ in two steps; ..; $(0.4)^{m-1}\cdot 0.6$ in m steps.
This is a geometric RV, and $P(\text{success in } m \geq 1 \text{ steps}) = 1$

If $X_0 = 2$, then
$$P(\text{transition to } 3) = 0.2\sum_{m=1}^{\infty}(0.5)^{m-1} = 0.4$$
If $X_0 = 3$, then $P(\text{transition of type } B) = 0$.

$\boxed{\text{Answer: } \pi_1 + \pi_2 \cdot 0.4 = 0.36}$

(d)
$$P(X_0 = 2 \mid D = \{1^{st} \text{ transition of type } B\}) = \frac{P(D \mid X_0 = 2)\cdot \pi_2}{P(D \mid X_0 = 1)\pi_1 + P(D \mid X_0 = 2)\pi_2}$$
$$= \frac{0.2\cdot 0.4}{0.6\cdot 0.2 + 0.2\cdot 0.4} = \frac{0.08}{0.2} = 0.4$$

6 a) $P(0,3) = e^{-3\lambda} = e^{-1.5}$

(b) $P(4 \le T \le 6) = P(0,4)(1 - P(0,2)) = e^{-4\lambda}(1 - e^{-2\lambda}) = e^{-2} - e^{-3}$

(c) We know that $ES = \frac{2}{\lambda} = 4$; $Var(S) = \frac{2}{\lambda^2} = 8$, so $ES^2 = Var(S) + (ES)^2 = 24$